Definitions:

- **Square roots of a**: The square root of a, denoted $\sqrt{a}$, is the number whose square is a. In other words,

$$\boxed{\sqrt{a} = b \quad \text{means} \quad b^2 = a.}$$

- **n-th roots of a**: The n-th root of a, denoted $\sqrt[n]{a}$, is a number whose n-th power equals a. In other words,

$$\boxed{\sqrt[n]{a} = b \quad \text{means} \quad b^n = a.}$$

  The number n is called the **index**.

Rules for n-th roots:

- **Product rule for radicals**: If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers and n is a positive integer,

$$\boxed{\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}.}$$

  In other words, the product of radicals is the radical of the product.

- **Quotient rule for radicals**: If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers and n is a positive integer,

$$\boxed{\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.}$$

  In other words, the radical of a quotient is the quotient of the radicals.

- **Index rule for radicals**: If m, n and k are positive integers, then

$$\boxed{\sqrt[kn]{a^{km}} = \sqrt[n]{a^m}}.$$

- If n is even, then $\sqrt[n]{a^n} = |a|$. For example, $\sqrt[4]{(-2)^4} = |-2| = 2$.
- If n is odd, then $\sqrt[n]{a^n} = a$. For example, $\sqrt[3]{(-6)^3} = -6$.

Simplifying radicals: A radical is in simplest form when the following conditions are satisfied.

- The quantity under the radical has no factor raised to a power greater than or equal to the index.
- There is no fraction under the radical.
- There is no radical in the denominator.
- There is no common factor, other than 1, between the exponents on factors under the radical and the index.

Important Properties:

- If $a \geq 0$, then $\sqrt{a}$ is the principal square root of a. If $a < 0$, then $\sqrt{a}$ cannot be evaluated in the real number system.
- If $a < 0$ and n is a positive even integer, then $\sqrt[n]{a}$ is not a real number.

Common Mistakes to Avoid:

- $\sqrt[n]{x+y} \neq \sqrt[n]{x} + \sqrt[n]{y}$.
- You may only use the Product (or Quotient) rule for radicals when the radicals have the same index.

PROBLEMS

Simplify each radical. Assume that all variables represent positive real numbers.:

1. $\sqrt{\dfrac{64}{81}}$

$$\sqrt{\frac{64}{81}} = \frac{\sqrt{64}}{\sqrt{81}} = \boxed{\frac{8}{9}}$$

---

2. $\sqrt[3]{\dfrac{x^9}{27}}$

$$\sqrt[3]{\frac{x^9}{27}} = \frac{\sqrt[3]{x^9}}{\sqrt[3]{27}} = \boxed{\frac{x^3}{3}}$$

---

3. $\sqrt[3]{-27x^3y^9z^6}$

$$\sqrt[3]{-27x^3y^9z^6} = \boxed{-3xy^3z^2}$$

---

4. $\sqrt[4]{16x^4y^{12}z^{16}}$

$$\sqrt[4]{16x^4y^{12}z^{16}} = \boxed{2xy^3z^4}$$

5. $\sqrt[3]{54x^3y^5z^4}$

$$\begin{aligned}\sqrt[3]{54x^3y^5z^4} &= \sqrt[3]{27 \cdot 2x^3y^3y^2z^3z} \\ &= \sqrt[3]{27x^3y^3z^3}\,\sqrt[3]{2y^2z} \\ &= \boxed{3xyz\,\sqrt[3]{2y^2z}}\end{aligned}$$

---

6. $\sqrt[4]{32x^5y^7z^9}$

$$\begin{aligned}\sqrt[4]{32x^5y^7z^9} &= \sqrt[4]{16 \cdot 2x^4xy^4y^3z^8z} \\ &= \sqrt[4]{16x^4y^4z^8}\,\sqrt[4]{2xy^3z} \\ &= \boxed{2xyz^2\,\sqrt[4]{2xy^3z}}\end{aligned}$$

---

7. $-\sqrt[5]{96x^7y^{19}z^{21}}$

$$\begin{aligned}-\sqrt[5]{96x^7y^{19}z^{21}} &= -\sqrt[5]{32 \cdot 3x^5x^2y^{15}y^4z^{20}z} \\ &= -\sqrt[5]{32x^5y^{15}z^{20}}\,\sqrt[5]{3x^2y^4z} \\ &= \boxed{-2xy^3z^4\,\sqrt[5]{3x^2y^4z}}\end{aligned}$$

---

8. $\sqrt[3]{128x^7y^2z^{19}}$

$$\begin{aligned}\sqrt[3]{128x^7y^2z^{19}} &= \sqrt[3]{64 \cdot 2x^6xy^2z^{18}z} \\ &= \sqrt[3]{64x^6z^{18}}\,\sqrt[3]{2xy^2z} \\ &= \boxed{4x^2z^6\,\sqrt[3]{2xy^2z}}\end{aligned}$$