Definition:

- **Rational exponent:** If m and n are positive integers with m/n in lowest terms, then

$$\boxed{a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m.}$$

(If n is even then we require $a \geq 0$.) In other words, in a rational exponent, the numerator indicates the power and the denominator indicates the root. For example,

$$8^{2/3} = \left(\sqrt[3]{8}\right)^2 = (2)^2 = 4.$$

Integer Exponent Rules:

- **Product Rule:** For any real numbers m and n,

$$\boxed{a^m \cdot a^n = a^{m+n}.}$$

When multiplying like bases, we add the exponents.

- **Quotient Rule:** For any nonzero number a and any real numbers m and n,

$$\boxed{\frac{a^m}{a^n} = a^{m-n}.}$$

When we divide like bases, we subtract the exponents.

- **Power Rule:** For any real numbers m and n,

$$\boxed{(a^m)^n = a^{mn}.}$$

When we raise a power to another power, we multiply the exponents.

- For any real number m,

$$\boxed{(ab)^m = a^m \cdot b^m.}$$

When we have a product raised to a power, we raise each factor to the power.

- For any real number m,

$$\boxed{\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}.}$$

When we have a quotient raised to a power, we raise both the numerator and denominator to the power.

- **Zero Exponent Rule:** For any nonzero real number a,

$$\boxed{a^0 = 1.}$$

- **Negative Exponents:** For any nonzero real number a and any real number n,

$$\boxed{a^{-n} = \frac{1}{a^n}.}$$

- For any nonzero numbers a and b, and any real numbers m and n,

$$\boxed{\frac{a^{-m}}{b^{-n}} = \frac{b^n}{a^m}.}$$

- For any nonzero numbers a and b, and any real numbers m and n,

$$\boxed{\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m.}$$

<u>**Common Mistakes to Avoid**</u>:

- When using the product rule, the bases MUST be the same. If they are not, then the expressions cannot be combined. Also, remember to keep the bases the same and only add the exponents. For example,

$$3^2 \cdot 3^4 = 3^{2+4} = 3^6 \qquad\qquad 3^2 \cdot 3^4 \neq 9^6.$$

- When using the quotient rule, the bases MUST be the same. If they are not, then the expressions cannot be combined. Also, remember to keep the bases the same and only subtract the exponents. For example,

$$\frac{4^5}{4^3} = 4^2 = 16.$$

- When using the power rule, remember that ALL factors are raised to the power. This includes any constants. For example,

$$\left(2x^3y^4\right)^2 = 2^2(x^3)^2(y^4)^2 = 4x^6y^8.$$

- A positive constant raised to a negative power does NOT yield a negative number. For example,

$$3^{-2} = \frac{1}{3^2} = \frac{1}{9}, \qquad\qquad 3^{-2} \neq -6.$$

- The Power Rule and Quotient Rule do NOT hold for sums and differences. In other words,

$$(a+b)^m \neq a^m + b^m \qquad \text{and} \qquad (a-b)^m \neq a^m - b^m.$$

- Note the difference between $(-16)^{1/4}$ which cannot be evaluated and $-16^{1/4} = -\sqrt[4]{16} = -2$. The placement of the negative sign does matter.
- Be careful when working with negative rational exponents.

$$a^{-1/n} \neq \frac{1}{a^n}.$$

Instead,

$$a^{-1/n} = \frac{1}{a^{1/n}}.$$

PROBLEMS

1. **Evaluate each number.**

(a) $16^{3/4}$

$$\begin{aligned} 16^{3/4} &= \left(\sqrt[4]{16}\right)^3 \\ &= (2)^3 \\ &= \boxed{8} \end{aligned}$$

(b) $\left(\frac{16}{81}\right)^{3/4}$

$$\begin{aligned} \left(\frac{16}{81}\right)^{3/4} &= \left(\sqrt[4]{\frac{16}{81}}\right)^3 \\ &= \left(\frac{2}{3}\right)^3 \\ &= \boxed{\frac{8}{27}} \end{aligned}$$

(c) $32^{-2/5}$

$$\begin{aligned} 32^{-2/5} &= \frac{1}{32^{2/5}} \\ &= \frac{1}{\left(\sqrt[5]{32}\right)^2} \\ &= \frac{1}{(2)^2} \\ &= \boxed{\frac{1}{4}} \end{aligned}$$

(d) $\left(\dfrac{25}{81}\right)^{-3/2}$

$$\begin{aligned} \left(\frac{25}{81}\right)^{-3/2} &= \left(\frac{81}{25}\right)^{3/2} \\ &= \left(\sqrt{\frac{81}{25}}\right)^3 \\ &= \left(\frac{9}{5}\right)^3 \\ &= \boxed{\frac{729}{125}} \end{aligned}$$

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2. **Use the rules of exponents to simplify each expression. Write all answers with positive exponents. Assume that all variable represent positive real numbers.**

(a) $\dfrac{(x^{2/3})^2}{(x^2)^{7/3}}$

$$\begin{aligned} \frac{(x^{2/3})^2}{(x^2)^{7/3}} &= \frac{x^{4/3}}{x^{14/3}} \\ &= x^{4/3-14/3} \\ &= x^{-10/3} \\ &= \boxed{\frac{1}{x^{10/3}}} \end{aligned}$$

(b) $\dfrac{x^{3/4}y^{1/2}}{(x^2y)^{1/4}}$

$$\begin{aligned}\frac{x^{3/4}y^{1/2}}{(x^2y)^{1/4}} &= \frac{x^{3/4}y^{1/2}}{x^{1/2}y^{1/4}} \\ &= x^{3/4-1/2}y^{1/2-1/4} \\ &= \boxed{x^{1/4}y^{1/4}}\end{aligned}$$

(c) $\left(2x^4y^{-4/5}\right)^3\left(8y^2\right)^{2/3}$

$$\begin{aligned}\left(2x^4y^{-4/5}\right)^3\left(8y^2\right)^{2/3} &= \left(8x^{12}y^{-12/5}\right)\left(8^{2/3}y^{4/3}\right) \\ &= \left(8x^{12}y^{-12/5}\right)\left(4y^{4/3}\right) \\ &= 32x^{12}y^{-12/5+4/3} \\ &= 32x^{12}y^{-36/15+20/15} \\ &= 32x^{12}y^{-16/15} \\ &= \boxed{\frac{32x^{12}}{y^{16/15}}}\end{aligned}$$

(d) $\dfrac{\left(x^{15}y^{-5}\right)^{1/5}}{\left(x^{-2}y^3\right)^{1/3}}$

$$\begin{aligned}\frac{\left(x^{15}y^{-5}\right)^{1/5}}{\left(x^{-2}y^3\right)^{1/3}} &= \frac{x^3y^{-1}}{x^{-2/3}y} \\ &= \frac{x^3x^{2/3}}{yy} \\ &= \frac{x^{3+2/3}}{y^2} \\ &= \boxed{\frac{x^{11/3}}{y^2}}\end{aligned}$$