FINITE NONSOLVABLE GROUPS WITH MANY DISTINCT CHARACTER DEGREES

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Abstract. Let G be a finite group and let $\mathrm{Irr}(G)$ denote the set of all complex irreducible characters of G. Let $\mathrm{cd}(G)$ be the set of all character degrees of G. For a degree $d \in \mathrm{cd}(G)$, the multiplicity of d in G, denoted by $m_G(d)$, is the number of irreducible characters of G having degree d. A finite group G is said to be a T_k-group for some integer $k \geq 1$ if there exists a nontrivial degree $d_0 \in \mathrm{cd}(G)$ such that $m_G(d_0) = k$ and that for every $d \in \mathrm{cd}(G) - \{1, d_0\}$, the multiplicity of d in G is trivial, that is, $m_G(d) = 1$. In this paper, we show that if G is a nonsolvable T_k-group for some integer $k \geq 1$, then $k = 2$ and $G \cong \mathrm{PSL}_2(5)$ or $\mathrm{PSL}_2(7)$.

1. Introduction

Let G be a finite group and let $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \cdots, \chi_k\}$ be the set of all complex irreducible characters of G. Let $\mathrm{cd}(G) = \{d_0, d_1, \cdots, d_t\}, 1 = d_0 < d_1 < \cdots < d_t$, be the set of all character degrees of G. For an integer $d \geq 1$, the multiplicity of d in G, denoted by $m_G(d)$, is the number of irreducible characters of G having the same degree d, i.e, $m_G(d) = |\{\chi \in \mathrm{Irr}(G) \mid \chi(1) = d\}|$. Let $n_i = \chi_i(1)$ for $1 \leq i \leq k$. We call $\mathrm{mp}(G) = (m_G(d_0), m_G(d_1), \cdots, m_G(d_t))$ the *multiplicity pattern* and $(n_1, n_2, \cdots, n_k)$ the *degree pattern* of G. Let $\mathbb{C}G$ be the complex group algebra of G. We know that $\mathbb{C}G = \bigoplus_{i=1}^{k} \mathrm{M}_{n_i}(\mathbb{C})$ and thus knowing the degree pattern of G is equivalent to knowing the structure of the complex group algebra of G; or equivalently the first column of the ordinary character table of G. One of the main questions in character theory of finite groups is Brauer's Problem 1 (see [6]) which asks for the possible degree patterns of finite groups. Recently, it has been proved in [19, 9] that the order of a finite group is bounded in terms of the largest multiplicity of its character degree. This gives a new restriction on the degree patterns of finite groups. Motivated by this result, we want to explore the relations between the multiplicities of character degrees of finite groups and the structure of the groups. In fact, this problem has already attracted many researchers in the literature (see [2, 3, 4, 11, 20]). In [20], G. Seitz classified all finite groups which have exactly one nonlinear irreducible representation. This result was generalized in [4] where the authors classified all finite groups G in which the multiplicity of every nonlinear irreducible character degree G is trivial. Also, the finite groups in which only two nonlinear irreducible characters have equal degrees have been classified in [2, 3]. To generalize these results, we consider the following definition. A finite

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group G is called a T_k-group for some integer $k \geq 1$ if there exists a nontrivial degree $d_0 \in \mathrm{cd}(G)$ with $m_G(d_0) = k$ and that for every nontrivial degree $d \in \mathrm{cd}(G)$ different from d_0, we have $m_G(d) = 1$. Obviously, the finite groups studied in [4, 20] and [2, 3] are exactly T_1-groups and T_2-groups, respectively. In this paper, we generalize results in [3] by proving.

Theorem A. *Let G be a finite nonsolvable group. If G is a T_k-group for some integer $k \geq 1$, then $G \cong \mathrm{PSL}_2(5)$ or $\mathrm{PSL}_2(7)$ and $k = 2$.*

Using [8], the multiplicity patterns of $\mathrm{PSL}_2(q)$ for $q \in \{5, 7\}$ are $(1, 2, 1, 1)$ and $(1, 2, 1, 1, 1)$, respectively. Suppose that G is a finite group such that $\mathrm{mp}(G) = \mathrm{mp}(\mathrm{PSL}_2(q))$ with $q \in \{5, 7\}$. As the first entry of $\mathrm{mp}(G)$ is $|G : G'|$ which is 1, we can see that G is perfect and, in particular, G is a nonsolvable T_2-group. Now by applying Theorem A, we deduce that $G \cong \mathrm{PSL}_2(5)$ or $\mathrm{PSL}_2(7)$. By comparing the number of distinct character degrees, we deduce that $G \cong \mathrm{PSL}_2(q)$. It follows that $\mathrm{PSL}_2(q)$ with $q \in \{5, 7\}$ are uniquely determined by the multiplicity patterns. In fact, it is conjectured in [23] that every nonabelian simple group is uniquely determined by the multiplicity pattern. Also, it has been shown in [23] that this conjecture holds for every nonabelian simple groups with at most 7 distinct character degrees. We note that this conjecture, if true, is a generalization of a result obtained in [24] where it is proved that all nonabelian simple groups are uniquely determined by the structure of their complex group algebras. This latter result is related to Brauer's Problem 2 which asks the following question: What does $\mathbb{C}G$ know about G? This is also an important question in character theory and has been studied extensively (see the references in [24]). Notice that if the degree pattern of a finite group G is given, then both $\mathrm{cd}(G)$ and $\mathrm{mp}(G)$ are known. Thus, apart from being a direct generalization of the results obtained in [3], Theorem A could be used to study questions raised in [23]. For finite solvable groups, if G is a finite T_k-group of odd order, then $|\mathrm{cd}(G)| \leq 2$ since G has only one real irreducible characters which is the trivial character and thus every nontrivial character degree of G has multiplicity at least 2. On the other hand, every finite group with exactly two distinct character degrees is a solvable T_k-group for some integer k (see [13, Corollary 12.6]) and a compete classification of such finite groups is yet to be found. This together with the fact that there is no explicit upper bound for k makes the classification of solvable T_k-groups quite complicated even for 2-groups.

Throughout this paper, all groups are finite and all characters are complex characters. Let G be a group. If $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$, then the inertia group of θ in G is denoted by $I_G(\theta)$. We write $\mathrm{Irr}(G|\theta)$ for the set of all irreducible constituents of θ^G. The order of an element $x \in G$ is denoted by $|x|$. Denote by $\Phi_k := \Phi_k(q)$ the value of the kth cyclotomic polynomial evaluated at q. Other notation is standard.

2. Preliminaries

In this section, we collect some results that will be needed in the next sections. We begin with the following easy lemma.

Lemma 2.1. *Let G be a group and let $N \trianglelefteq G$ such that G/N is cyclic of order $d \geq 2$. Assume that G has a nontrivial degree a with multiplicity m. Suppose that $a > d$ and $m/d \geq 2$. Then N has a nontrivial degree b with multiplicity at least 2 and $a/d \leq b \leq a$.*

Proof. Assume first that χ_N is not irreducible for some $\chi \in \mathrm{Irr}(G)$ with $\chi(1) = a$. Let $\theta \in \mathrm{Irr}(N)$ be an irreducible constituent of χ_N. As G/N is cyclic, we deduce from [13, Corollary 11.22] that θ is not G-invariant. Let $I = I_G(\theta)$ and $t := |G : I|$. Then $t \geq 2$. By Clifford's theorem [13, Theorem 6.2] and [13, Corollary 11.22] again, we deduce that $\chi_N = \sum_{i=1}^{t} \theta_i$, where $\theta_i \in \mathrm{Irr}(N)$ are distinct conjugates of θ. Hence N has a nontrivial degree a/t with multiplicity at least $t \geq 2$. Moreover, as $t \mid |G/N| = d$, we deduce that $a/d \leq a/t \leq a$. Assume now that $\chi_N \in \mathrm{Irr}(N)$ for every $\chi \in \mathrm{Irr}(G)$ with $\chi(1) = a$. It follows that N has a nontrivial degree a with multiplicity at least $m/d \geq 2$ as each irreducible character $\chi_N \in \mathrm{Irr}(N)$ has exactly d extensions in G. Therefore, in both cases N has a nontrivial degree b with $1 < a/d \leq b \leq a$ with multiplicity at least 2. The proof is now complete. □

We note that when d in the previous lemma is a prime, then $b \in \{a, a/d\}$. As an application of this lemma, we obtain the following corollary.

Corollary 2.2. *Let G be a group and let $N \trianglelefteq G$ such that G/N is cyclic of order $d \geq 2$. Assume that G has two nontrivial degrees $a_i, i = 1, 2$, with multiplicity $m_i, i = 1, 2$, respectively. Suppose that $a_2/d > a_1 > d$ and $m_i \geq 2d$ for $i = 1, 2$. Then N is not a T_k-group for any integer $k \geq 1$.*

Proof. By Lemma 2.1, for $i = 1, 2$, N has two nontrivial character degrees d_i such that $a_i/d \leq d_i \leq a_i$, each with multiplicity at least 2. Now we have $d_2 \geq a_2/d > a_1 \geq d_1 \geq a_1/d > 1$ by the hypothesis. Hence $d_i, i = 1, 2$ are nontrivial distinct character degrees of N and both degrees have nontrivial multiplicity, so N is not a T_k-group for any integer $k \geq 1$. □

The next result is well known. We refer to [7, 13.8, 13.9] for the notion of symbols and the classification of the unipotent characters of finite groups of Lie type.

Lemma 2.3. *Let S be a nonabelian simple group. Then the following hold:*

(1) *If S is a sporadic simple group, the Tits group or an alternating group of degree at least 7, then S has two nontrivial irreducible characters $\theta_i, i = 1, 2$ with distinct degree such that both θ_i extend to $\mathrm{Aut}(S)$.*
(2) *If S is a simple group of Lie type in characteristic p and $S \neq {}^2\mathrm{F}_4(2)'$, then the Steinberg character of S, denoted by St_S of degree $|S|_p$, is extendible to $\mathrm{Aut}(S)$. Furthermore, if $S \not\cong \mathrm{PSL}_2(3^f)$, then S possesses an irreducible character θ such that $\theta(1) \neq |S|_p$ and θ also extends to $\mathrm{Aut}(S)$.*

Proof. The first statement follows from Theorems 3 and 4 in [5]. For (2), the existence and extendibility of the Steinberg character of S is well known. Now assume that $S \not\cong \mathrm{PSL}_2(q)$ with $q = p^f$, then we can choose θ to be any unipotent character of S which is not one of the exceptions in [17, Theorem 2.5] and not the Steinberg character of S, then θ is extendible to $\mathrm{Aut}(S)$. (See [7, § 13.8, 13.9].) Finally, assume that $S \cong \mathrm{PSL}_2(q)$ with $q = p^f$ and $p \neq 3$. Then S has an irreducible character θ of degree $q + \delta$, where $q \equiv \delta \pmod 3$ and $\delta \in \{\pm 1\}$ such that θ extends to $\mathrm{Aut}(S)$. Notice that this irreducible character of S corresponds to a semisimple element of order 3 in the dual group $\mathrm{SL}_2(q)$. □

The following result due to Zsigmondy will be useful.

Lemma 2.4. (Zsigmondy [26]). *Let $q \geq 2$ and $n \geq 3$ be integers such that $(n, q) \neq (6, 2)$. Then $q^n - 1$ has a prime factor ℓ such that $\ell \equiv 1 \pmod n$ and ℓ does not divide $q^m - 1$ for any $m < n$.*

TABLE 1. Two tori for classical groups

$G = G(q)$	$\lvert T_1\rvert$	$\lvert T_2\rvert$	ℓ_1	ℓ_2
A_n	$(q^{n+1}-1)/(q-1)$	q^n-1	$\ell_{n+1}(q)$	$\ell_n(q)$
${}^2\mathrm{A}_n$, $(n \equiv 0(4))$	$(q^{n+1}+1)/(q+1)$	q^n-1	$\ell_{2n+2}(q)$	$\ell_n(q)$
${}^2\mathrm{A}_n$, $(n \equiv 1(4))$	$(q^{n+1}-1)/(q+1)$	q^n+1	$\ell_{(n+1)/2}(q)$	$\ell_{2n}(q)$
${}^2\mathrm{A}_n$, $(n \equiv 2(4))$	$(q^{n+1}+1)/(q+1)$	q^n-1	$\ell_{2n+2}(q)$	$\ell_{n/2}(q)$
${}^2\mathrm{A}_n$, $(n \equiv 3(4))$	$(q^{n+1}-1)/(q+1)$	q^n+1	$\ell_{n+1}(q)$	$\ell_{2n}(q)$
$\mathrm{B}_n, \mathrm{C}_n$ $(n \geq 3$ odd)	q^n+1	q^n-1	$\ell_{2n}(q)$	$\ell_n(q)$
$\mathrm{B}_n, \mathrm{C}_n$ $(n \geq 2$ even)	q^n+1	$(q^{n-1}+1)(q+1)$	$\ell_{2n}(q)$	$\ell_{2n-2}(q)$
D_n, $(n \geq 5$ odd)	$(q^{n-1}+1)(q+1)$	q^n-1	$\ell_{2n-2}(q)$	$\ell_n(q)$
D_n, $(n \geq 4$ even)	$(q^{n-1}+1)(q+1)$	$(q^{n-1}-1)(q-1)$	$\ell_{2n-2}(q)$	$\ell_{n-1}(q)$
${}^2\mathrm{D}_n$	q^n+1	$(q^{n-1}+1)(q-1)$	$\ell_{2n}(q)$	$\ell_{2n-2}(q)$

Such an ℓ is called a *primitive prime divisor* and is denoted by $\ell_n(q)$.

The orders of two maximal tori and the corresponding primitive prime divisors of the finite classical groups are given in Table 1. This is taken from [16, Table 3.5]. In Table 2, we list the degrees of some unipotent characters of the simple exceptional groups of Lie type. This can be found in [7, §13.9].

3. Simple T_k-groups

The main purpose of this section is to classify all simple T_k-group. As we will see shortly, there are only two simple T_k-groups and they are exactly the simple T_2-groups. Let $\mathscr{L}$ be the set consisting of the following simple groups:

$$\mathrm{PSL}_2(q), \mathrm{PSL}_3(q), \mathrm{PSU}_3(q), \mathrm{PSp}_4(q),$$
$$\mathrm{PSL}_6(2), \mathrm{PSL}_7(2), \mathrm{PSU}_4(2), \mathrm{PSp}_6(2), \mathrm{PSp}_8(2), \mathrm{P\Omega}_8^{\pm}(2)$$

and

$$\mathrm{PSL}_4(2), \mathrm{PSU}_4(3), \mathrm{PSU}_5(2), \mathrm{PSp}_6(3), \Omega_7(3),$$
$$\mathrm{PSp}_8(3), \Omega_9(3), \mathrm{P\Omega}_8^{\pm}(3), \mathrm{P\Omega}_{10}^{+}(2), \mathrm{P\Omega}_{10}^{-}(3).$$

The following result will be needed when dealing with simple classical groups of Lie type. We refer to [11, §4.3] and [14, Theorem 4.7] for some related results.

Lemma 3.1. *Let $\mathscr{G}$ be a simply connected simple algebraic group of classical type and let F be a suitable Frobenius map such that $S \cong \mathscr{G}^F/\mathrm{Z}(\mathscr{G}^F)$ is a simple classical group of Lie type defined over a finite field of size q with $S \notin \mathscr{L}$. Let the pair $(\mathscr{G}^*, F^*)$ be dual to $(\mathscr{G}, F)$ and let $G = (\mathscr{G}^*)^{F^*}$. For $i = 1, 2$, let T_i be the maximal tori of G with order given in Table 1. Then for each i, there exist two regular semisimple elements $s_i, t_i \in T_i$ such that $s_i, t_i \in T_i \cap G'$ and that s_i and t_i are not G-conjugate.*

Proof. Since $\mathscr{G}$ is of simply connected type, the dual group $\mathscr{G}^*$ is of adjoint type and thus by using the identifications with classical groups as given in [7, Page 40], G/S is either a cyclic or an elementary abelian group of order 4. In all cases, G/S is abelian and so $G' = S$. For each $i = 1, 2$, let $T_i' = T_i \cap G'$. Since $G' \trianglelefteq T_iG' \leq G$, we obtain that $|T_i'| = |T_i \cap G'| \geq |T_i|/d$ with $d := |G : G'|$. Since $S \notin \mathscr{L}$, both primitive prime divisors $\ell_i, i = 1, 2$ in Table 1 exist.

Claim 1: For $i = 1, 2$, every element $s_i \in T_i$ of order ℓ_i is a regular semisimple element and $s_i \in T_i \cap G' = T_i'$.

Observe that the two maximal tori of G with order given in Table 1 have the properties that they are uniquely determined up to conjugation by their orders. Furthermore, for each $i = 1, 2$, the conjugacy class of maximal tori containing T_i is the only class of maximal tori whose order is divisible by ℓ_i. Also, the Sylow ℓ_i-subgroups of G are cyclic. Let $s_i \in T_i$ be a semisimple element of order ℓ_i. As in the proof of [18, Proposition 2.4], if $\mathrm{C}_G(s_i)$ is not a torus, then its semisimple rank is at least 1, and thus it contains two maximal tori of different orders. Both of these tori must have orders divisible by ℓ_i, which is impossible. Hence we obtain that $\mathrm{C}_G(s_i) = T_i$. Since $\gcd(\ell_i, |G : G'|) = 1$, we deduce that $s_i \in G'$ and so $s_i \in T_i' = T_i \cap G'$.

Claim 2: For every $x \in T_i$, if $|x|$ is divisible by ℓ_i, then x is a regular semisimple element.

Assume that $x \in T_i$ such that $|x| = m\ell_i$ where $m \geq 1$ is an integer. Let $s = x^m \in T_i$. Then $|s| = \ell_i$ and so by Claim 1, we know that $\mathrm{C}_G(s) = T_i$, hence $T_i \leq \mathrm{C}_G(x) \leq \mathrm{C}_G(x^m) = \mathrm{C}_G(s) = T_i$. So, $\mathrm{C}_G(x) = T_i$ and x is a regular semisimple element.

Claim 3: If $T_i' = T_i \cap G'$ has no element whose order is a proper multiple of ℓ_i, then T_i' contains two distinct G-conjugacy classes of regular semisimple elements of order ℓ_i.

For $i = 1, 2$, let T_i'' be a cyclic subgroup of T_i' whose order is divisible by ℓ_i. By our assumption, we must have that $|T_i''| = \ell_i$ and so $T_i'' = \langle s_i \rangle$, where s_i is an element of order ℓ_i. As s_i is regular semisimple by Claim 1, we deduce that $\mathrm{N}_G(\langle s_i \rangle) \leq \mathrm{N}_G(T_i)$ and so as the Sylow ℓ_i-subgroup of T_i is cyclic and $|s_i| = \ell_i$, we obtain that $\mathrm{N}_G(\langle s_i \rangle) = \mathrm{N}_G(T_i)$. Since $\mathrm{C}_G(s_i) = T_i = \mathrm{C}_G(T_i)$ and $|\mathrm{N}_G(T_i)/T_i| \leq m(S)$, we deduce that $|\mathrm{N}_G(\langle s_i \rangle)/\mathrm{C}_G(s_i)| \leq m(S)$, where $m(S)$ is the dimension of the natural module for $S = G'$ over $\mathbb{F}_q$. (Notice that the fact $|\mathrm{N}_G(T_i)/T_i| \leq m(S)$ can be deduced from [1, Lemma 4.7].) It follows that s_i is G-conjugate to at most $m(S)$ of its powers and thus $T_i'' = \langle s_i \rangle$ contains at least $\varphi(|s_i|)/m(S)$ G-conjugacy classes of regular semisimple elements of order ℓ_i, where φ is the Euler φ-function. Since $|s_i| = \ell_i$, we deduce that $\varphi(|s_i|)/m(S) = (|\ell_i| - 1)/m(S) \geq (|T_i''| - 1)/m(S)$. We now verify that for each possibility of S, we have that $(|T_i''| - 1)/m(S) \geq 2$, which implies that T_i' contains at least two distinct G-conjugacy classes of regular semisimple elements of order ℓ_i.

(a) Assume first that $S \cong \mathrm{PSL}_n(q)$. Then $m(S) = n$ and $d = \gcd(n, q-1)$. Since $T_i, i = 1, 2$ are cyclic, we deduce that both T_i' are also cyclic of order at least $|T_i|/d$. Hence we can choose $T_i'' = T_i'$ for $i = 1, 2$. Then $|T_1''| \geq (q^n - 1)/(d(q-1))$ and $|T_2''| \geq (q^{n-1} - 1)/d$. As $S \notin \mathscr{L}$, it is routine to check that $q^{n-1} - 1 \geq (2n+1)(q-1)$ and so since $q - 1 \geq d = \gcd(n, q-1)$, we obtain that $q^{n-1} - 1 \geq (2n+1)d$ or equivalently $(|T_2''| - 1)/n \geq 2$. Similarly, we can check that $(|T_1''| - 1)/n \geq 2$.

(b) Assume that $S \cong \mathrm{PSU}_n(q)$ and $n \geq 5$ is odd. We have that $m(S) = n$ and $d = \gcd(n, q+1)$. As in the previous case, we see that both T_i are cyclic and so are T_i', hence we can choose $T_i'' = T_i'$. Then $|T_1''| \geq (q^n + 1)/(d(q+1))$ and $|T_2''| \geq (q^{n-1} - 1)/d$. Since $q^{n-1} - 1 > (q^n + 1)/(q+1)$ and $d = \gcd(n, q+1) \leq q+1$, it suffices to show that $(q^n + 1) \geq (2n+1)(q+1)^2$ with $n \geq 5$ odd. Since $S \notin \mathscr{L}$, we can check that the previous inequality holds so that $(|T_i''| - 1)/m(S) \geq 2$ for $i = 1, 2$, as required.

(c) Assume that $S \cong \mathrm{PSU}_n(q)$ and $n \geq 4$ is even. Arguing as in the case n is odd, we have that $|T_1''| \geq (q^n-1)/(d(q+1))$ and $|T_2''| \geq (q^{n-1}+1)/d$. Since $q^{n-1}+1 > (q^n-1)/(q+1)$ and $d \leq q+1$, it suffices to show that $q^n - 1 \geq (2n+1)(q+1)^2$ where $n \geq 4$ is even. As $S \notin \mathscr{L}$, we can check that the previous inequality holds so that $(q^n-1)/(n(q+1)^2) \geq 2$ and thus $(|T_i''|-1)/m(S) \geq 2$ for $i = 1, 2$, as required.

(d) Assume that $S \cong \mathrm{PSp}_{2n}(q)$ or $\Omega_{2n+1}(q)$ and $n \geq 3$ is odd. We have that $m(S) \leq 2n+1$ and $d = \gcd(2, q-1)$. In this case both T_i are cyclic, so we can choose $T_i'' = T_i'$ and hence $|T_1''| \geq (q^n+1)/d$ and $|T_2''| \geq (q^n-1)/d$. Since $q^n+1 > q^n-1$, it suffices to show that $q^n - 1 \geq (4n+3)d$, where $n \geq 3$ is odd and $d = \gcd(2, q-1)$. Since $S \notin \mathscr{L}$, we can check that the latter inequality holds, so for $i = 1, 2$, we obtain that $(|T_i''|-1)/m(S) \geq 2$.

(e) Assume that $S \cong \mathrm{PSp}_{2n}(q)$ or $\Omega_{2n+1}(q)$ and $n \geq 4$ is even. We can choose $|T_1''| = |T_1'| \geq (q^n+1)/d$ and $|T_2''| \geq (q^{n-1}+1)/d$. Since $q^n+1 > q^{n-1}+1$, it suffices to show that $q^{n-1}+1 \geq (4n+3)d$, where $n \geq 4$ is even and $d = \gcd(2, q-1)$. Since $S \notin \mathscr{L}$, we can check that the latter inequality holds, so for $i = 1, 2$, we have that $(|T_i''|-1)/m(S) \geq 2$.

(f) Assume that $S \cong P\Omega_{2n}^+(q)$ where $n \geq 5$ is odd. Then $m(S) = 2n$ and $d = \gcd(4, q^n-1)$. We have $|T_1''| \geq (q^{n-1}+1)/d$ and $|T_2''| \geq (q^n-1)/d$. Since $q^n - 1 > q^{n-1}+1$, it suffices to show that $q^{n-1}+1 \geq (4n+1)d$, where $n \geq 5$ is odd. Since $S \notin \mathscr{L}$, we can check that the latter inequality holds, so for $i = 1, 2$, we obtain that $(|T_i''|-1)/m(S) \geq 2$.

(g) Assume that $S \cong P\Omega_{2n}^+(q)$ where $n \geq 4$ is even. Then $|T_1''| \geq (q^{n-1}+1)/d$ and $|T_2''| \geq (q^{n-1}-1)/d$. Since $q^{n-1}+1 > q^{n-1}-1$, it suffices to show that $q^{n-1}-1 \geq (4n+1)d$, where $n \geq 4$ is even. Since $S \notin \mathscr{L}$, we can check that the latter inequality holds, so for $i = 1, 2$, $(|T_i''|-1)/m(S) \geq 2$.

(h) Assume that $S \cong P\Omega_{2n}^-(q)$ where $n \geq 4$. Then $|T_1''| \geq (q^n+1)/d$ and $|T_2''| \geq (q^{n-1}+1)/d$. Since $q^n+1 > q^{n-1}+1$, it suffices to show that $q^{n-1}+1 \geq (4n+1)d$, where $n \geq 4$. Since $S \notin \mathscr{L}$, we can check that the latter inequality holds, so for $i = 1, 2$, $(|T_i''|-1)/m(S) \geq 2$ as wanted. This completes the proof of Claim 3.

Finally, by Claim 1, to finish the proof of the lemma, we only need to find a regular simisimple element $t_i \in T_i'$ such that t_i is not G-conjugate to s_i for $i = 1, 2$. Now, for each i, if T_i' contains an element whose order is a proper multiple of ℓ_i, then this element is a regular semisimple element by Claim 2 and clearly it is not G-conjugate to s_i as the orders of these two semisimple elements are distinct. Otherwise, if no such elements exists, then by Claim 3 we can find a regular semisimple element $t_i \in T_i'$ with the same order as that of s_i and they are not G-conjugate. The proof is now complete. □

We now prove the main result of this section.

Theorem 3.2. *Let S be a nonabelian simple group. If S is a T_k-group for some integer $k \geq 1$, then $k = 2$ and $S \cong \mathrm{PSL}_2(5)$ or $\mathrm{PSL}_2(7)$.*

Proof. Using the classification of finite simple groups, we consider the following cases:

(1) S is a sporadic simple group or the Tits group. It is routine to check using [8] that S has at least two nontrivial distinct degrees, each with multiplicity at least 2. Hence S is not a T_k-group for any integer $k \geq 1$.

(2) $S \cong \mathrm{A}_n$ with $n \geq 5$. If $n = 5$, then $\mathrm{cd}(\mathrm{A}_5) = \{1, 3, 4, 5\}$ and every degree of A_5 has multiplicity 1, except for the degree 3 with multiplicity 2. Hence A_5 is

TABLE 2. Some unipotent characters of simple exceptional groups of Lie type

$S = S(q)$	Symbol	Degree
${}^2\mathrm{G}_2(q^2)$	cuspidal	$\frac{1}{\sqrt{3}}q\Phi_1\Phi_2\Phi_4$
	cuspidal	$\frac{1}{2\sqrt{3}}q\Phi_1\Phi_2\Phi'_{12}$
${}^2\mathrm{F}_4(q^2)$	$({}^2\mathrm{B}_2[a],1), ({}^2\mathrm{B}_2[b],1)$	$\frac{1}{\sqrt{2}}q\Phi_1\Phi_2\Phi_4^2\Phi_6$
	$({}^2\mathrm{B}_2[a],\epsilon), ({}^2\mathrm{B}_2[b],\epsilon)$	$\frac{1}{\sqrt{2}}q^{13}\Phi_1\Phi_2\Phi_4^2\Phi_6$
$\mathrm{G}_2(q)$	$\phi'_{1,3}, \phi''_{1,3}$	$\frac{1}{3}q\Phi_3\Phi_6$
	$\mathrm{G}_2[\theta], \mathrm{G}_2[\theta^2]$	$\frac{1}{3}q\Phi_1^2\Phi_2^2$
$\mathrm{F}_4(q)$	$\phi'_{8,3}, \phi''_{8,3}$	$q^3\Phi_4^2\Phi_8\Phi_{12}$
	$\phi'_{8,9}, \phi''_{8,9}$	$q^9\Phi_4^2\Phi_8\Phi_{12}$
${}^2\mathrm{E}_6(q)$	${}^2\mathrm{E}_6[\theta], {}^2\mathrm{E}_6[\theta^2]$	$\frac{1}{3}q^7\Phi_1^4\Phi_2^6\Phi_4^2\Phi_8\Phi_{10}$
$\mathrm{E}_6(q)$	$\mathrm{E}_6[\theta], \mathrm{E}_6[\theta^2]$	$\frac{1}{3}q^7\Phi_1^6\Phi_2^4\Phi_4^2\Phi_5\Phi_8$
$\mathrm{E}_7(q)$	$\mathrm{E}_7[\xi], \mathrm{E}_7[-\xi]$	$\frac{1}{2}q^{11}\Phi_1^7\Phi_3^3\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{12}$
	$\mathrm{E}_6[\theta], \mathrm{E}_6[\theta^2]$	$\frac{1}{3}q^7\Phi_1^6\Phi_2^6\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_{10}\Phi_{14}$
$\mathrm{E}_8(q)$	$(\mathrm{E}_7[\xi],1), (\mathrm{E}_7[-\xi],1)$	$\frac{1}{2}q^{11}\Phi_1^7\Phi_3^4\Phi_4^4\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{20}\Phi_{24}$
	$(\mathrm{E}_7[\xi],\epsilon), (\mathrm{E}_7[-\xi],\epsilon)$	$\frac{1}{2}q^{26}\Phi_1^7\Phi_3^4\Phi_4^4\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{20}\Phi_{24}$

a T_2-group. Now assume that $n \geq 6$. For $6 \leq n \leq 13$, we can check that A_n is not a T_k-group by using [8]. Thus we can assume that $n \geq 14$. Let λ be a self-conjugate partition of n and denote by χ^λ the irreducible character of S_n labeled by λ. Then χ^λ when restricted to A_n will split into the sum of two irreducible characters having the same degree $\chi^\lambda(1)/2$. Thus $\chi^\lambda(1)/2 \in \mathrm{cd}(A_n)$ has multiplicity at least two. Therefore, in order to show that A_n is not a T_k-group for any $k \geq 1$, it suffices to find two distinct self-conjugate partitions $\lambda_i, i = 1, 2$, of n such that $\chi^{\lambda_i}(1)/2, i = 1, 2$ are distinct and nontrivial.

Assume first that $n \geq 15$ is odd. We can write $n = 2k + 9 = 2(k + 4) + 1$. Then $\lambda_1 = (k+5, 1^{k+4})$ and $\lambda_2 = (k+3, 3^2, 1^k)$ are two distinct self-conjugate partitions of n. Assume next that $n \geq 14$ is even. Write $n = 2k+8$. Then $\lambda_1 = (k+4, 2, 1^{k+2})$ and $\lambda_2 = (k + 3, 3^2, 1^k)$ are two distinct self-conjugate partitions of n. Using Hook formula, we can easily check that $\chi^{\lambda_i}(1)/2$ are distinct and nontrivial for $i = 1, 2$. Thus A_n is not a T_k-group for $n \geq 14$.

(3) S is a simple exceptional group of Lie type in characteristic p.

Assume first that $S \cong {}^2\mathrm{B}_2(q^2)$, where $q^2 = 2^{2m+1}$ and $m \geq 1$. By [22], S has irreducible characters of degree $\sqrt{2}q(q^2 - 1)/2$ and $q^4 + 1$, with multiplicity 2 and $(q^2 - 2)/2$, respectively. Hence S is not a T_k-group for any $k \geq 1$.

Assume next that $S \cong {}^3\mathrm{D}_4(q)$. If $q = 2$, then ${}^3\mathrm{D}_4(2)$ is not a T_k-group for any integer $k \geq 1$ by using [8]. Hence we can assume that $q \geq 3$. By [10, Table 4.4], S has degrees $(q^3 + \delta)(q^2 - \delta q + 1)(q^4 - q^2 + 1)$ with multiplicity $\frac{1}{2}q(q + \delta)$, where $\delta = \pm 1$. Since $q \geq 3$, we deduce that these two degrees are distinct and nontrivial and $q(q+1)/2 \geq q(q-1)/2 \geq 3$, so S is not a T_k-group.

Assume that $S \cong \mathrm{E}_6(q)$ and let $G = \mathrm{E}_6(q)_{ad}$. Let $d = |G : S| = \gcd(3, q-1)$. By [15], G has an irreducible character χ of degree

$$\chi(1) = \frac{1}{2}q^3\Phi_1^2\Phi_3^2\Phi_4\Phi_5\Phi_6^2\Phi_8\Phi_9\Phi_{12} \text{ with } m_G(\chi(1)) \geq q(q-1).$$

Obviously, $\chi(1) > d$ and $m_G(\chi(1)) \geq 2d$. By Lemma 2.1, S has a nontrivial degree $b \in \{\chi(1), \chi(1)/d\}$ with nontrivial multiplicity. By Table 2, S also has a nontrivial degree $\psi(1)$ with multiplicity at least 2. Observe that $\psi(1) \notin \{\chi(1), \chi(1)/d\}$. Therefore, S has two distinct nontrivial degrees, each with multiplicity at least 2, so S is a not a T_k-group.

The same argument applies to the simple group $S \cong {}^2\mathrm{E}_6(q)$ with $q > 2$ since $G = {}^2\mathrm{E}_6(q)_{ad}$ has a degree

$$\chi(1) = \frac{1}{2}q^3\Phi_2^2\Phi_3\Phi_4\Phi_6^3\Phi_8\Phi_{10}\Phi_{12}\Phi_{18} \text{ with } m_G(\chi(1)) \geq (q+1)(q-2),$$

$|G : S| = \gcd(3, q+1) =: d$; and S has a nontrivial degree $\psi(1) \notin \{\chi(1), \chi(1)/d\}$ with multiplicity at least 2 by Table 2. For the case $q = 2$, we can check that ${}^2\mathrm{E}_6(2)$ is not a T_k-group by using [8].

Finally, for the remaining simple exceptional groups of Lie type, by Table 2 each simple group S has two distinct nontrivial degrees, each with multiplicity at least 2, so S is not a T_k-group.

(4) Assume that S is a simple classical group in characteristic p.

(4_1) Assume first that $S \notin \mathscr{L}$. We consider the following set up. Let $\mathscr{G}$ be a simply connected simple algebraic group of classical type and let F be a suitable Frobenius map such that $L/Z(L) \cong S$, where $L = \mathscr{G}^F$. Let the pair $(\mathscr{G}^*, F^*)$ be dual to $(\mathscr{G}, F)$ and let $G = (\mathscr{G}^*)^{F^*}$. Let $T \leq G$ be a maximal torus of G. By Deligne-Lusztig's theory, for each G-conjugacy class of regular semisimple element $s \in T$, there exists a simisimple character $\chi_s \in \mathrm{Irr}(L)$ with degree $|G : T|_{p'}$ and if $s \in G'$, then $\mathrm{Z}(L) \subseteq \ker\chi_s$, so χ_s is an irreducible character of $L/\mathrm{Z}(L) \cong S$. Moreover, if $t \in T \cap G'$ is also a regular semisimple element which is not G-conjugate to s, then the semisimple character $\chi_t \in \mathrm{Irr}(L)$ is an irreducible character of S with the same degree as that of χ_s and thus the nontrivial degree $|G : T|_{p'} \in \mathrm{cd}(S)$ has multiplicity at least 2.

Since $S \notin \mathscr{L}$, by Lemma 3.1 G contains two maximal tori $T_i, i = 1, 2$ such that each $T_i' = T_i \cap G'$ possesses two regular semisimple elements s_i and t_i which are not G-conjugate. By the discussion above, we deduce that each nontrivial degree $|G : T_i|_{p'} \in \mathrm{cd}(S)$ has multiplicity at least 2. Since $|G : T_i|_{p'}, i = 1, 2$ are distinct and nontrivial, we deduce that S is not a T_k-group for any integer $k \geq 1$.

(4_2) Assume next that $S \in \mathscr{L}$.

(a) Assume first that $S \cong \mathrm{PSL}_2(q)$ with $q \geq 4$. As $\mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5) \cong \mathrm{A}_5$, we can assume that $q \geq 7$. If $q = 7$, then $\mathrm{PSL}_2(7)$ is a T_2-group by using [8]. Hence we assume that $q \geq 8$. If q is even, then $S \cong \mathrm{SL}_2(q)$ has degrees $q-1$ and $q+1$ with multiplicity $q/2$ and $q/2-1$, respectively. Since $q \geq 8$, we can see that $q/2 > q/2 - 1 \geq 3$, so S is not a T_k-group. Now assume that $q \geq 9$ is odd. Since $\mathrm{PSL}_2(9) \cong \mathrm{A}_6$, we can assume that $q \geq 11$. We know that S has two irreducible characters of degree $(q+\epsilon)/2$ where $q \equiv \epsilon \pmod 4$ and $\epsilon \in \{1, -1\}$. Furthermore, S has a nontrivial degree $q-1$ with multiplicity $(q-\delta)/4$, where $q \equiv \delta \pmod 4$ and $\delta \in \{1, 3\}$. As $(q-1)/4 > (q-3)/4 \geq (11-3)/4 = 2$ and $q-1 > (q+1)/2 \geq (q-1)/2$, S has two distinct nontrivial degrees, each with multiplicity at least 2 and thus S is not a T_k-group.

(b) Assume that $S \cong \mathrm{PSL}_3(q)$. Since $\mathrm{PSL}_3(2) \cong \mathrm{PSL}_2(7)$, we can assume that $q \geq 3$. For $3 \leq q \leq 11$, we can check that $\mathrm{PSL}_3(q)$ is not a T_k-group by using [8]. So, we assume that $q \geq 13$. In this case, by [21] S has degrees $d_1 = q^2 + q + 1$ and $d_2 = q(q^2 + q + 1)$, both with multiplicity $(q-1)/d - 1$. Since $q \geq 13$ and

$d = \gcd(3, q-1) \leq 3$, it follows that $(q-1)/d - 1 \geq (q-1)/3 - 1 \geq 2$ and hence S is not a T_k-group in these cases.

(c) Assume that $S \cong \mathrm{PSU}_3(q)$. Since $\mathrm{PSU}_3(2)$ is not simple, we can assume that $q \geq 3$. For $3 \leq q \leq 9$, we can check that $\mathrm{PSU}_3(q)$ is not a T_k-group by using [8]. So, we assume that $q \geq 11$. In this case, by [21] S has degrees $d_1 = q^2 - q + 1$ and $d_2 = q(q^2 - q + 1)$, both with multiplicity $(q+1)/d - 1$. Since $q \geq 11$ and $d = \gcd(3, q+1) \leq 3$, it follows that $(q+1)/d - 1 \geq (q+1)/3 - 1 \geq 2$. Thus S is not a T_k-group.

(d) Assume that $S \cong \mathrm{PSp}_4(q)$ with $q \geq 3$. If $q \geq 4$ is even, then S possesses two distinct nontrivial degrees $q(q^2+1)/2$ and $(q-1)(q^2+1)$ with multiplicity 2 and q respectively by using [15]. Now assume that $q \geq 3$ is odd. Using [15] again, G has two distinct nontrivial character degrees $a_1 := 2q(q^2+1)/2$ and $a_2 := (q+1)(q^2+1)$ with multiplicity 4 and $3(q-3)/2$, respectively. Since $d = |G : S| = 2$, we deduce that $a_2/d > a_1 > d$. If $q \geq 5$, then $3(q-3)/2 \geq 2d = 4$ and $4 \geq 2d$, so it follows from Corollary 2.2 that S is not a T_k-group. For the remaining cases, we can check directly using [8] that S is not a T_k-group.

(e) Finally, for the remaining simple groups in $\mathscr{L}$, it is routine to check using [12] that S is not a T_k-group for any $k \geq 1$. The proof is now complete. $\square$

4. Nonsolvable T_k-groups

We first prove a special case of the main theorem. In fact, we show that non-perfect T_k-groups must be solvable. We note that if G is a T_k-group for some integer $k \geq 1$ and $N \trianglelefteq G$, then since $\mathrm{Irr}(G/N) \subseteq \mathrm{Irr}(G)$, we can easily see that G/N is also a T_m-group for some integer $m \leq k$.

Theorem 4.1. *If G is a non-perfect T_k-group for some $k \geq 1$, then G is solvable.*

Proof. Let G be a counterexample to the theorem with minimal order. Then $G' \neq G$ and G is a T_k-group for some $k \geq 1$ but G is nonsolvable. Let M be the last term of the derived series of G and let $N \trianglelefteq G$ such that M/N is a chief factor of G. Since G is nonsolvable, we see that M is nontrivial and hence it is perfect, so M/N is nonabelian and $M/N \cong W^t$ for some nonabelian simple group W and some integer $t \geq 1$. Then M/N is a minimal normal subgroup of G/N and that

$$|G/N : (G/N)'| = |G/N : G'/N| = |G : G'| > 1$$

as G is non-perfect and $N \leq M \leq G'$. It follows that G/N is a non-perfect nonsolvable group and since $\mathrm{Irr}(G/N) \subseteq \mathrm{Irr}(G)$, we deduce that G/N is a non-perfect nonsolvable T_m-group for some integer $m \geq 1$. If N is nontrivial, then $|G/N| < |G|$, which contradicts the minimality of $|G|$. Therefore, we conclude that N must be trivial and $M \cong W^t$.

Claim 1: $M \cong \mathrm{PSL}_2(3^f)$ for some $f \geq 2$.

We first show that $W \cong \mathrm{PSL}_2(3^f)$ with $f \geq 2$. Suppose by contradiction that $W \ncong \mathrm{PSL}_2(3^f)$ with $f \geq 2$. Then there exist two irreducible characters $\theta_i \in \mathrm{Irr}(W)$ such that $\theta_1(1) \neq \theta_2(1)$ and both θ_i extend to $\mathrm{Aut}(W)$ by Lemma 2.3. Let $\varphi_i = \theta_i^t \in \mathrm{Irr}(M)$ for $i = 1, 2$. By [5, Lemma 5], we deduce that both φ_i extend to $\chi_i \in \mathrm{Irr}(G)$. Furthermore, by Gallagher's theorem [13, Corollary 6.17] we know that each φ_i has exactly $|G/M : (G/M)'| = |G : G'|$ extensions. For each i, all extensions of φ_i have the same degree which is $\varphi_i(1) = \theta_i^t(1) > 1$. So, G has two distinct nontrivial degrees $\theta_i^t(1), i = 1, 2$, both with nontrivial multiplicity, which is a contradiction. Hence $W \cong \mathrm{PSL}_2(3^f)$ with $f \geq 2$ as we wanted.

We now claim that $t = 1$ and thus $M \cong \mathrm{PSL}_2(3^f)$ with $f \geq 2$. By way of contradiction, assume that $t \geq 2$. Let θ be the Steinberg character of W. Then $\varphi = \theta^t \in \mathrm{Irr}(M)$ extends to $\varphi_0 \in \mathrm{Irr}(G)$ by [5, Lemma 5]. Thus by Gallagher's theorem [13, Corollary 6.17] again, we have that $\varphi_0(1) = \varphi(1) = \theta(1)^t$ is a nontrivial degree with nontrivial multiplicity. It follows that if $d \in \mathrm{cd}(G)$ with $1 < d \neq \theta(1)^t = 3^{tf}$, then the multiplicity of d is trivial. It is well know that $\mathrm{PSL}_2(3^f)$ has two irreducible characters of degree $(3^f+\epsilon)/2$, where $3^f \equiv \epsilon \pmod 4$ and $\epsilon \in \{\pm 1\}$, hence $\mathrm{PSL}_2(3^f)$ has a nontrivial degree $(3^f+\epsilon)/2 < 3^f$ with multiplicity 2. Denote these two irreducible characters by $\alpha_i, i = 1, 2$. For $i = 1, 2$, let

$$\varphi_i = 1 \times 1 \times \cdots \times \alpha_i \in \mathrm{Irr}(M)$$

and

$$\psi = 1 \times 1 \times \cdots \times \theta \in \mathrm{Irr}(M).$$

Let I, I_1 and I_2 be the inertia groups of ψ, φ_1 and φ_2, respectively. Obviously, we have that $M \leq I_i \leq I \leq G$ for $i = 1, 2$. By the representation theory of wreath products, we know that ψ extends to $\psi_0 \in \mathrm{Irr}(I)$ and $|G : I| = t$. Since G/M is solvable, we deduce that I/M is solvable. If I/M is nontrivial, then I/M has $j > 1$ linear characters and thus ψ has j distinct extensions to I, which are $\psi_0\lambda$ with $\lambda \in \mathrm{Irr}(I/M)$ and $\lambda(1) = 1$, so by Clifford's theorem [13, Theorem 6.11] we have that $(\psi_0\lambda)^G \in \mathrm{Irr}(G)$ are distinct irreducible characters of G having the same degree. Furthermore, for $\lambda \in \mathrm{Irr}(I/M)$ with $\lambda(1) = 1$, we have

$$(\psi_0\lambda)^G(1) = \psi_0^G(1) = |G : I|\psi_0(1) = 3^f \cdot t < 3^{tf}.$$

The last inequality holds since $t \geq 2$. But then this is a contradiction since the multiplicity of the nontrivial degree $3^f \cdot t$ is at least $|I/M : (I/M)'|$ which is nontrivial by our assumption. Therefore, we conclude that I/M is trivial and so $I = M$. It follows that for $i = 1, 2$, we have $I_i = I = M$ since $M \leq I_i \leq I = M$. Thus for each i, we have $\varphi_i^G \in \mathrm{Irr}(G)$ and

$$\varphi_i^G(1) = |G : M|\alpha_i(1) = |G : M|(3^f + \epsilon)/2$$

which is nontrivial and different from 3^{ft}. Clearly, $\varphi_1^G \neq \varphi_2^G$, so we deduce that G has a nontrivial degree $|G : M|(3^f + \epsilon)/2 \neq 3^{ft}$ with multiplicity at least 2, which is impossible. This contradiction proves our claim.

Claim 2: G is an almost simple group with socle M.

By the previous claim, we know that $M \cong \mathrm{PSL}_2(3^f)$ with $f \geq 2$. Let $C = C_G(M)$. Then $C \trianglelefteq G$ and G/C is an almost simple group with socle MC/C. Assume first that G/C is perfect. Then $G = MC$ and since M is nonabelian simple, we must have that $M \cap C = 1$ and so $G = M \times C$, where $G/M \cong C$ is solvable. If C is nontrivial, then $|C : C'| > 1$ and so for each nontrivial irreducible character $\mu \in \mathrm{Irr}(M)$ of M, we see that μ has $|C : C'|$ extensions to $G = M \times C$ and thus G cannot be a T_k-group for any $k \geq 1$. Hence C must be trivial and so G is simple, which is impossible as $G \neq G'$. Assume next that G/C is non-perfect. Then G/C is a non-perfect nonsolvable T_m-group for some $m \geq 1$. By the minimality of $|G|$, we must have that $C = 1$ and thus G is an almost simple group with socle $M \cong \mathrm{PSL}_2(3^f)$.

The Final Contradiction. Let $q = 3^f$, with $f \geq 2$. Let α be an irreducible character of M with $\alpha(1) = (q + \epsilon)/2$ where $q \equiv \epsilon \pmod 4$ and $\epsilon \in \{\pm 1\}$, let δ be the diagonal automorphism of M and let φ be the field automorphism of M of order f. Then $\mathrm{Out}(M) = \langle \delta \rangle \times \langle \varphi \rangle$. Since G is non-perfect, we deduce that

$|G:M|$ is nontrivial. Observe that the Steinberg character St_M of M is extendible to $\mathrm{Aut}(M)$ and so it extends to G by [5, Lemma 5], hence by Gallagher's theorem [13, Corollary 6.17] the degree $\mathrm{St}_M(1)=|M|_3$ has multiplicity at least $|G:M|>1$. Thus the multiplicity of every nontrivial degree of G different from $|M|_3=q$ must be trivial. By [25, Lemma 4.6], α is φ-invariant. Now if $G\leq M\langle\varphi\rangle$, then α is G-invariant and since G/M is cyclic, α extends to G and so G has a nontrivial degree $(q+\epsilon)/2$ with multiplicity at least $|G:M|\geq 2$, which is impossible. Hence $G\not\leq M\langle\varphi\rangle$. By [25, Theorem 6.5], we have $I_G(\alpha)=G\cap M\langle\varphi\rangle$ and $|G:I_G(\alpha)|=2$. If $|I_G(\alpha):M|>1$, then α has $|I_G(\alpha):M|$ extensions to $I_G(\alpha)$ as $I_G(\alpha)/M$ is cyclic and thus by inducing these characters to G, we see that G has at least $|I_G(\alpha):M|\geq 2$ irreducible characters of degree $q+\epsilon$, which is a contradiction. Thus, we conclude that $I_G(\alpha)=M$ and $|G:M|=2$. By [25, Corollary 6.2], we have that either $G\cong\mathrm{PGL}_2(q)$ or $G\cong M\langle\delta\varphi^{f/2}\rangle$, where f is even. Clearly, the first case cannot happen. For the latter case, since f is even, we obtain that $q\equiv 1 \pmod 4$, so M has exactly $(q-1)/4$ irreducible characters of degree $q-1$. As $|G:M|=2$, by [25, Theorem 6.6] all irreducible characters of G lying over an irreducible character of M of degree $q-1$ have degree $2(q-1)$. Therefore, G has at least $(q-1)/8$ irreducible characters of degree $2(q-1)$. If $q=9$, then we can check directly using [8] that all almost simple groups with socle $\mathrm{PSL}_2(9)\cong\mathrm{A}_6$ are not T_k-groups for any integer $k\geq 1$. Thus we can assume that $q\geq 81$ as f is even, so $(q-1)/8\geq 10$, hence G is not a T_k-group. This final contradiction proves our theorem. $\square$

We are now ready to prove the main theorem.

Proof of Theorem A. Let G be a counterexample to the theorem with minimal order. Then G is a nonsolvable T_k-group for some integer $k\geq 1$ but G is isomorphic to neither $\mathrm{PSL}_2(5)$ nor $\mathrm{PSL}_2(7)$. If $G'\neq G$, then G is solvable by Theorem 4.1, which is a contradiction. Thus we can assume that G is perfect. Let M be a maximal normal subgroup of G. Then G/M is a nonabelian simple group. Since $\mathrm{Irr}(G/M)\subseteq\mathrm{Irr}(G)$, we deduce that G/M is a T_m-group for some $m\geq 1$. Now by Theorem 3.2, we have that $G/M\cong\mathrm{PSL}_2(q)$ with $q\in\{5,7\}$ and G/M is a T_2-group. Since G is a counterexample to the theorem, we deduce that M is nontrivial.

By [8], we know that $\mathrm{cd}(\mathrm{PSL}_2(5))=\{1,3,4,5\}$ with multiplicity $1,2,1,1$ and $\mathrm{cd}(\mathrm{PSL}_2(7))=\{1,3,6,7,8\}$ with multiplicity $1,2,1,1,1$. Since G is a T_k-group, we deduce that the degree $3\in\mathrm{cd}(G/M)$ is the unique nontrivial character degree in $\mathrm{cd}(G)$ with nontrivial multiplicity and also $k\geq 2$. If $k=2$, then G must be isomorphic to either $\mathrm{PSL}_2(5)$ or $\mathrm{PSL}_2(7)$ by [3, Main Theorem], which is a contradiction. Therefore, we must have that $k\geq 3$. Hence there exists $\chi\in\mathrm{Irr}(G|M)$ with $\chi(1)=3$ and thus $\chi\in\mathrm{Irr}(G|\theta)$ for some nontrivial irreducible character θ of M. By Clifford's theorem [13, Theorem 6.2], we have that $\chi_M=e(\theta_1+\theta_2+\cdots+\theta_t)$, where all θ_i are conjugate to θ in G, $e\geq 1$ is the degree of an irreducible projective representation of $I_G(\theta)/M$ and $t=|G:I_G(\theta)|$. Since $\chi(1)=3$, we have that $3=et\theta(1)$ and hence $t\leq 3$. As the index of a proper subgroup of G/M with $G/M\cong\mathrm{PSL}_2(5)$ or $\mathrm{PSL}_2(7)$ is at least 5, we must have that $t=1$, so $\chi_M=e\theta$ and θ is G-invariant. If θ is extendible to $\theta_0\in\mathrm{Irr}(G)$, then $\theta_0(1)=\theta(1)\geq 2$ since G is perfect; also by Gallagher's theorem [13, Corollary 6.17], we obtain that $\mathrm{Irr}(G|\theta)=\{\theta_0\lambda|\lambda\in\mathrm{Irr}(G/M)\}$. It follows that $\chi=\mu\theta_0$ for some $\mu\in\mathrm{Irr}(G/M)$. As $3=\chi(1)=\mu(1)\theta(1)$ and $\theta(1)\geq 2$, we must have that $\mu(1)=1$ and $\theta(1)=3$. Since $3\in\mathrm{cd}(G/M)$ has multiplicity 2, there exist two distinct irreducible characters $\lambda_i, i = 1, 2$ of G/M with $\lambda_i(1) = 3$ and so $\theta_0\lambda_i, i = 1, 2$ are two distinct irreducible characters of G, both have degree 9. Thus $9 \in \mathrm{cd}(G)$ has multiplicity at least 2 which is a contradiction as $3 \in \mathrm{cd}(G)$ already has nontrivial multiplicity. Thus θ is G-invariant but it is not extendible to G. As the Schur multiplier of $\mathrm{PSL}_2(q)$ with $q \in \{5, 7\}$ is cyclic of order 2, by the theory of character triple isomorphism [13, Chapter 11], the triple (G, M, θ) must be isomorphic to $(\mathrm{SL}_2(q), A, \lambda)$, where $q \in \{5, 7\}$, $A = \mathrm{Z}(\mathrm{SL}_2(q))$ and μ is a nontrivial irreducible character of A. Since

$$\mathrm{cd}(\mathrm{SL}_2(q)|\lambda) = \{(q - \epsilon)/2, q - 1, q + 1\},$$

with $q \equiv \epsilon \pmod 4$ and $\epsilon \in \{\pm 1\}$, we deduce that

$$\mathrm{cd}(G|\theta) = \{\theta(1)(q - \epsilon)/2, (q - 1)\theta(1), (q + 1)\theta(1)\}.$$

However, we can check that all degrees in $\mathrm{cd}(G|\theta)$ are even and thus $3 \notin \mathrm{cd}(G|\theta)$. This contradiction proves the theorem. □

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