SYMMETRIC GROUPS ARE DETERMINED BY THEIR CHARACTER DEGREES

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ABSTRACT. Let G be a finite group. Let $X_1(G)$ be the first column of the ordinary character table of G. In this paper, we will show that if $X_1(G) = X_1(S_n)$, then $G \cong S_n$. As a consequence, we show that S_n is uniquely determined by the structure of the complex group algebra $\mathbb{C}S_n$.

1. Introduction and Notation

All groups considered are finite and all characters are complex characters. Let G be a group and let $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \cdots, \chi_k\}$ be the set of all irreducible characters of G. Put $n_i = \chi_i(1)$. We say that $(n_1, n_2, \cdots, n_k)$ is the *degree pattern* of G. Let $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$ be the set of all irreducible character degrees of G. Following [2], let $X_1(G)$ be the first column of the ordinary character table of G. By a suitable re-ordering of the rows in the character table of G, we can see that $X_1(G)$ coincides with the degree pattern $(n_1, n_2, \cdots, n_k)$ of G. We also consider $X_1(G)$ as a multiset consisting of character degrees of G counting multiplicities. Since $|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2$, the order of G is known given $X_1(G)$. There are examples showing that non-isomorphic groups may have the same character table and so the first column of their character tables coincide. Using the classification of finite simple groups, it is easy to see that non-abelian simple groups are uniquely determined by their character table. It was shown by Nagao [14] that the symmetric groups S_n are also uniquely determined by their character tables. In [16], we know that the alternating group A_n of degree at least 5, and the sporadic simple groups are uniquely determined by the first column of their character tables. In this paper, we will prove a similar result for the symmetric groups.

Theorem 1.1. *Let G be a finite group. If $X_1(G) = X_1(S_n)$, then $G \cong S_n$.*

This gives a positive answer to [2, Question 126]. Let $\mathbb{C}$ be the complex number field and let G be a group. Denote by $\mathbb{C}G$ the group algebra of G over $\mathbb{C}$. Let $G_i, i = 1, 2$, be groups. By Molien's Theorem ([2, Theorem 2.13]) we know that $\mathbb{C}G_1 \cong \mathbb{C}G_2$ if and only if $X_1(G_1) = X_1(G_2)$. Therefore, knowing the first column of the character table of a group G is equivalent to knowing the structure of the group algebra $\mathbb{C}G$. It is known that $\mathbb{C}G$ allows us to recognize the Frobenius groups or the p-nilpotent groups ([2, Corollaries 10.11, and 10.27]). Now Theorem 1.1 yields.

Corollary 1.2. *Let G be a group. If $\mathbb{C}G \cong \mathbb{C}S_n$, then $G \cong S_n$.*

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We should mention that Brauer's Problem 1 (see [3]) which asks the following: What are the possible degree patterns of finite groups? Little is known about this problem. Now Corollary 1.2 says that there is exactly one isomorphism type of the group algebra with a degree pattern as that of the symmetric groups.

We now outline our argument for the proof of Theorem 1.1. Assume that $X_1(G) = X_1(S_n)$. We first observe that $|G : G'| = 2, |G| = n!, k(G) = k(S_n)$ and $\mathrm{cd}(G) = \mathrm{cd}(S_n)$, where $k(G)$ denotes the number of conjugacy classes of G. The result is trivial when $n \leq 4$. Hence we will assume that $n \geq 5$. Next we will show that G' is perfect, that is $G' = G''$, by applying [8, Lemma 12.3]. Choose $M \leq G'$ be a normal subgroup of G so that G'/M is a chief factor of G. As $|G : G'| = 2$ and G'/M is non-abelian, we deduce that $G'/M \cong S^k$, where S is a non-abelian simple group and k is at most 2. We proceed to show that G'/M must be a simple group that is $k = 1$. This is done by applying Theorem 3.3. We now deduce that either G/M is an almost simple group with socle G'/M or $G/M \cong G'/M \times \mathbb{Z}_2$. We now apply Theorem 3.1 which asserts that if H is an almost simple group and $\mathrm{cd}(H) \subseteq \mathrm{cd}(S_n), n \geq 5$, then the socle of H must be isomorphic to A_n, to show that $G'/M \cong A_n$. Assume that $n \neq 6$. By comparing the orders, $G \cong S_n$ or $G \cong A_n \times \mathbb{Z}_2$. Finally, using the fact that G and S_n have the same number of irreducible characters, we can eliminate the latter case. Thus G must be isomorphic to S_n. In the exceptional case, we have $|\mathrm{Out}(A_6)| = 4$. In this case, G is one of the following groups: $A_6 \times \mathbb{Z}_2, PGL_2(9) \cong A_6.2_2, M_{10} \cong A_6.2_3$ or S_6. Using [5], we conclude that $G \cong S_6$. We remark that this argument is based on the Huppert's method given in [7]. This method is used to verify the Huppert Conjecture which states that non-abelian simple groups are determined by their sets of character degrees (see [7, 17]).

If $\mathrm{cd}(G) = \{s_0, s_1, \cdots, s_t\}$, where $1 = s_0 < s_1 < \cdots < s_t$, then we define $d_i(G) = s_i$ for all $1 \leq i \leq t$. Then $d_i(G)$ is the i^{th} smallest degree of the non-trivial character degrees of G. If n is an integer then we denote by $\pi(n)$ the set of all prime divisors of n. If G is a group, we will write $\pi(G)$ instead of $\pi(|G|)$ to denote the set of all prime divisors of the order of G. Let $p(G) = max(\pi(G))$ be the largest prime divisor of the order of G and let $\rho(G) = \cup_{\chi \in \mathrm{Irr}(G)} \pi(\chi(1))$ be the set of all primes which divide some irreducible character degrees of G. Finally, if $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$, then the inertia group of θ in G is denoted by $I_G(\theta)$. Other notation is standard.

2. Preliminaries

Let n be a positive integer. We call $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ a partition of n, written $\lambda \vdash n$, provided $\lambda_i, i = 1, 2, \ldots, r$ are integers, with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$ and $\sum_{i=1}^r \lambda_i = n$. We collect the same parts together and write $\lambda = (\ell_1^{a_1}, \ell_2^{a_2}, \cdots, \ell_k^{a_k})$, with $\ell_i > \ell_{i+1} > 0$ for $i = 1, \cdots, k-1; a_i \neq 0$; and $\sum_{i=1}^k a_i \ell_i = n$. It is well known that the irreducible complex characters of the symmetric group S_n are parametrized by partitions of n. Denote by χ^λ the irreducible character of S_n corresponding to partition λ. The irreducible characters of the alternating group A_n are then obtained by restricting χ^λ to A_n. In fact, χ^λ is still irreducible upon restriction to the alternating group A_n if and only if λ is not self-conjugate. Otherwise, χ^λ splits into two irreducible characters of A_n having the same degree. The following result on the minimal character degrees of symmetric groups is due to Rasala [15].

Lemma 2.1. ([15, Result 3]). *Let λ be a partition of n.*

(*a*) *If* $n \geq 15$, *then the first* 6 *nontrivial minimal character degrees of* S_n *are:*
(1) $d_1(S_n) = n-1$ *and* $\lambda \in \{(n-1,1),(2,1^{n-2})\}$;
(2) $d_2(S_n) = \frac{1}{2}n(n-3)$ *and* $\lambda \in \{(n-2,2),(2^2,1^{n-4})\}$;
(3) $d_3(S_n) = d_2(S_n)+1 = \frac{1}{2}(n-1)(n-2)$ *and* $\lambda \in \{(n-2,1^2),(3,1^{n-3})\}$;
(4) $d_4(S_n) = \frac{1}{6}n(n-1)(n-5)$ *and* $\lambda \in \{(n-3,3),(2^3,1^{n-6})\}$;
(5) $d_5(S_n) = \frac{1}{6}(n-1)(n-2)(n-3)$ *and* $\lambda \in \{(n-3,1^3),(4,1^{n-4})\}$;
(6) $d_6(S_n) = \frac{1}{3}n(n-2)(n-4)$ *and* $\lambda \in \{(n-3,2,1),(3,2,1^{n-5})\}$;
(*b*) *If* $n \geq 22$, *then the next five smallest character degrees are:*
(7) $d_7(S_n) = n(n-1)(n-2)(n-7)/24$ *and* $\lambda \in \{(n-4,4),(2^4,1^{n-8})\}$;
(8) $d_8(S_n) = (n-1)(n-2)(n-3)(n-4)/24$ *and* $\lambda \in \{(n-4,1^4),(5,1^{n-5})\}$;
(9) $d_9(S_n) = n(n-1)(n-4)(n-5)/12$ *and* $\lambda \in \{(n-4,2^2),(3^2,1^{n-6})\}$;
(10) $d_{10}(S_n) = n(n-1)(n-3)(n-6)/8$ *and* $\lambda \in \{(n-4,3,1),(3,2^2,1^{n-7})\}$;
(11) $d_{11}(S_n) = n(n-2)(n-3)(n-5)/8$ *and* $\lambda \in \{(n-4,2,1^2),(4,2,1^{n-6})\}$;

Assume $n \geq 5$. Using [5, 6] and Lemma 2.1, we can see that $d_1(S_n) = n-1$ and $d_2(S_n) = n(n-3)/2$ if $n \neq 8$ while $d_2(S_8) = 14$. Similarly, if $n \geq 6$, then $d_1(A_n) = n-1$ while $d_1(A_5) = 3$ (see [16]). The following is well-known.

Lemma 2.2. (Tschebyschef). *If* $m \geq 15$, *then there is at least one prime* p *with* $m/2 < p \leq m$.

Proof. If $m \geq 17$ then the result follows from [11, Proposition 5.1]. For $15 \leq m \leq 16$, the lemma is obvious. □

The following results on the classification of prime power character degrees of symmetric groups will be used frequently.

Lemma 2.3. ([1, Theorem 5.1]). *Suppose that* S_n *possesses a non-trivial irreducible character* χ *with* $\chi(1) = p^d$, *where* p *is a prime. Then one of the following holds:*
(1) $n = p^d+1$, *and* $\chi(1) = p^d$;
(2) $n = 4$ *and* $\chi(1) = 2$;
(3) $n = 5$ *and* $\chi(1) = 5$;
(4) $n = 6$ *and* $\chi(1) \in \{3^2, 2^4\}$;
(5) $n = 8$ *and* $\chi(1) = 2^6$;
(5) $n = 9$ *and* $\chi(1) = 3^3$;

We refer to [4, 13.8, 13.9] for the classification of unipotent characters and the notion of symbols.

Lemma 2.4. *Let* S *be a simple group of Lie type in characteristic* p *defined over a field of size* q. *Assume that* $S \neq L_2(q), {}^2F_4(2)'$. *Then there exist two irreducible characters* $\chi_i, i = 1,2$, *of* S *such that both* χ_i *extend to* $\mathrm{Aut}(S)$ *with* $1 < \chi_1(1) < \chi_2(1)$ *and* $\chi_2(1) = |S|_p$. *In particular, if* G *is an almost simple group with socle* S, *where* $S \neq L_2(q), {}^2F_4(2)'$, *then* $|S|_p > d_1(G)$.

Proof. By the results of Lusztig [9], any unipotent character θ of S has an extension $\tilde{\theta}$ to the group G_1 of inner-diagonal automorphisms of S such that θ and $\tilde{\theta}$ have the same inertia group in $\mathrm{Aut}(S)$ (see [10, Proposition 2.1]). Moreover, the unipotent characters of G_1 remain irreducible upon restriction to S, and these restrictions are all the unipotent characters of S. By [10, Theorem 2.4], all unipotent characters of S extend to their inertia groups in $\mathrm{Aut}(S)$. By results of Lusztig, the inertia group of a unipotent character of S is exactly $\mathrm{Aut}(S)$ except for several cases explicitly listed in [10, Theorem 2.5]. Thus we can choose χ_2 to be the Steinberg character of S and χ_1 to be any unipotent character of S such that χ_1 does not appear in [10, Theorem 2.5] and $1 < \chi_1(1) < \chi_2(1) = |S|_p$.

Assume S is of type A_{n-1} with $n \geq 3$. We have $G_1 = (A_{n-1})_{ad}(q) = PGL_n(q)$. By [4, 13.8], the unipotent characters of G_1 are parametrized by partitions of n. Let $\alpha = (1, n-1)$. Then the degree of the unipotent character χ^α corresponding to α is given by $\chi^\alpha(1) = (q^n - q)/(q-1)$. Since $St_S(1) = |S|_p = q^{n(n-1)/2}$, and $n \geq 3$, we have $St_S(1) > \chi^\alpha(1) > 1$.

Assume S is of type ${}^2A_{n-1}$, where $n \geq 3$. Then $G_1 = ({}^2A_{n-1})_{ad}(q^2) = PU_n(q)$. By [4, 13.8], the unipotent characters of G_1 are again parametrized by partitions of n. Let $\alpha = (1, n-1)$. Then the degree of the unipotent character χ^α corresponding to α is given by $\chi^\alpha(1) = (q^n + (-1)^n q)/(q+1)$. Since $St_S(1) = |S|_p = q^{n(n-1)/2}$, and $n \geq 3$, we have $St_S(1) > \chi^\alpha(1) > 1$.

Assume next that S is of type B_n, or C_n where $n \geq 2$ and $S \neq S_4(2)$. Then $G_1 = (B_n)_{ad}(q) = SO_{2n+1}(q)$ or $G_1 = (C_n)_{ad}(q) = PCSp_{2n}(q)$. By [4, 13.8], G_1 has a unipotent characters χ^α labeled by the symbol

$$\alpha = \begin{pmatrix} 0 & 1 & n \\ & - & \end{pmatrix}$$

with $\chi^\alpha(1) = (q^n - 1)(q^n - q)/(2(q+1))$. Since $|S|_p = q^{n^2}$ and $(n, q) \neq (2, 2)$, we see that $|S|_p > \chi^\alpha(1) > 1$.

Assume S is of type $D_n(q), n \geq 4$. Then $G_1 = (D_n)_{ad}(q) = P(CO_{2n}(q)^0)$. By [4, 13.8], G_1 has a unipotent character χ^α labeled by the symbol

$$\alpha = \begin{pmatrix} n-1 \\ 1 \end{pmatrix}$$

with $\chi^\alpha(1) = (q^n - 1)(q^{n-1} + q)/(q^2 - 1)$. Since $|S|_p = q^{n(n-1)}$, we see that $|S|_p > \chi^\alpha(1) > 1$.

Assume S is of type ${}^2D_n(q^2), n \geq 4$. Then $G_1 = ({}^2D_n)_{ad}(q^2) = P(CO^-_{2n}(q)^0)$ and G_1 has a unipotent character χ^α labeled by the symbol

$$\alpha = \begin{pmatrix} 1 & & n-1 \\ & - & \end{pmatrix}$$

with $\chi^\alpha(1) = (q^n + 1)(q^{n-1} - q)/(q^2 - 1)$. Since $|S|_p = q^{n(n-1)}$, $|S|_p > \chi^\alpha(1) > 1$.

For the simple groups of exceptional type, we will use the explicit list of unipotent characters in [4, 13.9].

Assume S is of type $G_2(q)$. Then S has a unipotent character labeled by $\theta_{2,1}$ with degree $q\Phi_2^2\Phi_3/6$. As $G_2(2) \cong U_3(3).2$ is not simple, we can assume that $q \geq 3$. Since $|S|_p = q^6$, we have $q\Phi_2^2\Phi_3/6 < q^6$ so that $1 < \theta_{2,1}(1) < |S|_p$.

Assume S is of type ${}^3D_4(q^3)$. Then S has a unipotent character labeled by $\theta_{1,3'}$ with degree $q\Phi_{12}$. Since $|S|_p = q^{12}$, we have $1 < \theta_{1,3'}(1) < |S|_p$.

Assume S is of type $F_4(q)$. Then S has a unipotent character labeled by $\theta_{9,2}$ with degree $q^2\Phi_3^2\Phi_6^2\Phi_{12}$. Since $|S|_p = q^{24}$, we have $1 < \theta_{9,2}(1) < |S|_p$.

Assume S is of type $E_6(q)$. Then S has a unipotent character labeled by $\theta_{6,1}$ with degree $q\Phi_8\Phi_9$. Since $|S|_p = q^{36}$, we have $1 < \theta_{6,1}(1) < |S|_p$.

Assume S is of type ${}^2E_6(q^2)$. Then S has a unipotent character labeled by $\theta_{2,4'}$ with degree $q\Phi_8\Phi_{18}$. Since $|S|_p = q^{36}$, we have $1 < \theta_{2,4'}(1) < |S|_p$.

Assume S is of type $E_7(q)$. Then S has a unipotent character labeled by $\theta_{7,1}$ with degree $q\Phi_7\Phi_{12}\Phi_{14}$. Since $|S|_p = q^{63}$, we have $1 < \theta_{7,1}(1) < |S|_p$.

Assume S is of type $E_8(q)$. Then S has a unipotent character labeled by $\theta_{8,1}$ with degree $q\Phi_4^2\Phi_8\Phi_{12}\Phi_{20}\Phi_{24}$. Since $|S|_p = q^{120}$, we have $1 < \theta_{8,1}(1) < |S|_p$.

Assume S is of type ${}^2B_2(q^2)$, where $q^2 = 2^{2m+1}$ and $m \geq 1$. Then S has a unipotent character labeled by ${}^2B_2[a]$ with degree $q\Phi_1\Phi_2/\sqrt{2}$. Since $|S|_p = q^4$, we have $1 < {}^2B_2[a](1) < |S|_p$.

Assume S is of type ${}^2G_2(q^2)$, where $q^2 = 3^{2m+1}$ and $m \geq 1$. Then S has a unipotent character θ with degree $q\Phi_1\Phi_2\Phi_4/\sqrt{3}$. Since $|S|_p = q^6$, we have $1 < \theta(1) < |S|_p$.

Assume S is of type ${}^2F_4(q^2)$, where $q^2 = 2^{2m+1}$ and $m \geq 1$. Then S has a unipotent character labeled by ${}^2B_2[a]$ with degree $q\Phi_1\Phi_2\Phi_4^2\Phi_6/\sqrt{2}$. Since $|S|_p = q^{24}$, we have $1 < {}^2B_2[a](1) < |S|_p$. This finishes the proof of the first assertion.

Now assume G is an almost simple group with socle S, where $S \neq L_2(q), {}^2F_4(2)'$. Let $\chi_i \in \mathrm{Irr}(S), i = 1, 2$, be irreducible characters of S obtained above. As both χ_i extend to $\mathrm{Aut}(S)$ and $S \trianglelefteq G \leq \mathrm{Aut}(S)$, we deduce that each χ_i extends to $\psi_i \in \mathrm{Irr}(G)$ with $\psi_i(1) = \chi_i(1)$ for $i = 1, 2$. Thus $\psi_2(1) = |S|_p > \psi_1(1) = \chi_1(1) > 1$ so that $|S|_p > d_1(G)$ as required. The proof is now complete. □

Lemma 2.5. *Let G be an almost simple group with socle $S = L_2(q)$, where $q = p^f \geq 7$. If $p \neq 3$, then G contains an irreducible character of degree $q + \delta$ where $q \equiv \delta \pmod 3$. If $p = 3$, then G contains an irreducible character of degree $(q+\epsilon)/2$ or $q + \epsilon$, where $q \equiv \epsilon \pmod 4$.*

Proof. Assume first that $p \neq 3$. It follows from the proof of [13, Proposition 3.7] that the irreducible character of S corresponding to a semisimple element of order 3 in the dual group $SL_2(q)$, of degree $q + \delta$, where $q \equiv \delta \pmod 3$, is extendible to $\mathrm{Aut}(S)$, and hence G contains an irreducible character of degree $q + \delta$ as required. Note that this result fails for $L_2(3^f)$. Now we assume that $q = 3^f$. Observe that $L_2(q)$ always contains irreducible characters χ_a, χ_b of degree $q - 1$ and $q + 1$, respectively, which are extendible to $PGL_2(q)$. Thus if $L_2(q) \trianglelefteq G \leq PGL_2(q)$ then G possesses characters of degree $q \pm 1$. Now assume that $G \nleq PGL_2(q)$. Recall that the only outer automorphisms of $L_2(q)$ are the diagonal automorphisms and the field automorphisms. It is well-known that S contains two irreducible characters $\chi^{\pm}$ of degree $(q + \epsilon)/2$, where $q \equiv \epsilon \pmod 4$. Now the diagonal automorphisms of S fuse these two irreducible characters while the field automorphisms fix those two. Let $\theta \in \{\chi^+, \chi^-\}$. Then $\theta \in \mathrm{Irr}(S)$ and $I_{\mathrm{Aut}(S)}(\theta) = P\Gamma L_2(q)$. Thus if $S \trianglelefteq G \leq P\Gamma L_2(q)$, then θ is G-invariant and so θ extends to G as G/S is cyclic. Hence G has an irreducible character of degree $(q + \epsilon)/2$. Finally, assume $PGL_2(q) \trianglelefteq G \leq \mathrm{Aut}(L_2(q))$. Then the irreducible character μ of $PGL_2(q)$ lying over θ is of degree $q + \epsilon$. We see that μ is G-invariant and hence it extends to G, as $G/PGL_2(q)$ is cyclic. Therefore G contains an irreducible character of degree $q + \epsilon$. The proof is now complete. □

Corollary 2.6. *If G is an almost simple group then $\rho(G) = \pi(G)$.*

Proof. Observe first that for any $\chi \in \mathrm{Irr}(G)$, we have $\chi(1)$ divides $|G|$ by [8, Theorem 3.11]. Hence $\rho(G) \subseteq \pi(G)$. As G is almost simple, it has no normal abelian Sylow p-subgroup, so that by the Ito-Michler Theorem [12], every prime divisor of G must divide $\chi(1)$ for some $\chi \in \mathrm{Irr}(G)$, and thus $\pi(G) \subseteq \rho(G)$. Hence $\rho(G) = \pi(G)$ as required. □

Lemma 2.7. *Let G and H be groups. Suppose that $\mathrm{cd}(G) \subseteq \mathrm{cd}(H)$. Then*

Table 1. Sporadic simple groups and their automorphism groups

S	$p(S)$	$d_1(S)$	$d_2(S)$	$d_3(S)$
M_{11}	11	10	11	16
M_{12}	11	11	16	45
$M_{12}.2$	11	22	32	45
J_1	19	56	76	77
M_{22}	11	21	45	55
$M_{22}.2$	11	21	45	55
J_2	7	14	21	36
$J_2.2$	7	28	36	42
M_{23}	23	22	45	230
HS	11	22	77	154
$HS.2$	11	22	77	154
J_3	19	85	323	324
$J_3.2$	19	170	324	646
M_{24}	23	23	45	231
McL	11	22	231	252
$McL.2$	11	22	231	252
He	17	51	153	680
$He.2$	17	102	306	680
Ru	29	378	406	783
Suz	13	143	364	780
$Suz.2$	13	143	364	780
$O'N$	31	10944	13376	25916
$O'N.2$	31	10944	26752	37696
Co_3	23	23	253	275
Co_2	23	23	253	275
Fi_{22}	13	78	429	1001
$Fi_{22}.2$	13	78	429	1001
HN	19	133	760	3344
$HN.2$	19	266	760	3344
Ly	67	2480	45694	48174
Th	31	248	4123	27000
Fi_{23}	23	782	3588	5083
Co_1	23	276	299	1771
J_4	43	1333	299367	887778
Fi'_{24}	29	8671	57477	249458
$Fi'_{24}.2$	29	8671	57477	249458
B	47	4371	96255	1139374
M	71	196883	21296876	842609326
${}^2F_4(2)'$	13	26	27	78
${}^2F_4(2)'.2$	13	27	52	78

(i) $d_i(G) \geq d_i(H)$, *for all* $i \geq 1$;
(ii) If G *is almost simple then* $\pi(G) \subseteq \pi(H)$.

Proof. (i) is obvious. (ii) follows from Corollary 2.6 as $\rho(G) \subseteq \rho(H) \subseteq \pi(H)$. □

3. Proofs of the main results

Theorem 3.1. *Let G be an almost simple group with socle S and let $n \geq 5$ be an integer. If $\mathrm{cd}(G) \subseteq \mathrm{cd}(S_n)$ then $S \cong A_n$.*

Proof. Using the classification of finite simple groups, S is an alternating group of degree at least 5, a finite simple group of Lie type or one of the 26 sporadic groups. We will treat the Tits group as a sporadic group rather than a group of Lie type.

Step 1. Eliminate simple groups of Lie type. By way of contradiction, we assume that S is a simple group of Lie type in characteristic p and $\mathrm{cd}(G) \subseteq \mathrm{cd}(S_n)$ with $n \geq 5$. By the isomorphisms $L_2(4) \cong L_2(5) \cong A_5, L_2(9) \cong A_6$ and $L_4(2) \cong A_8$, we can assume that S is not one of the groups listed above nor the Tits group. It is well known that the Steinberg character of S of degree $|S|_p$ extends to $\chi \in \mathrm{Irr}(G)$ and hence $\chi(1) = |S|_p$ is a non-trivial power of p. Assume first that $|S|_p$ is not the minimal character degree of S_n, that is $|S|_p > n - 1$. It follows from Lemma 2.3 that $n \in \{5, 6, 8, 9\}$ and $|S|_p = 5, |S|_p \in \{3^2, 2^3\}, |S|_p = 2^6, |S|_p = 3^3$, respectively. It is routine to check that these cases cannot happen. Thus $|S|_p = n - 1 = d_1(S_n)$. Now Lemma 2.4 will provide a contradiction unless $S = L_2(q)$, where $q = p^f \geq 4$. Assume that $S = L_2(q)$ and thus $q \geq 7$. As $|S|_p = q = d_1(S_n)$, by Lemma 2.5, G must contain an irreducible character of degree $q + 1$ and hence $q + 1 \in \mathrm{cd}(S_n)$. Since $S \neq L_4(2) \cong A_8$, we have $d_2(S_n) = n(n-3)/2$. We have $n - 1 = q$ and hence as $q \geq 7$, we obtain $d_2(S_n) = n(n-3)/2 = (q+1)(q-2)/2 > q+1 > q = d_1(S_n)$, which contradicts Lemma 2.7(i). This finishes the proof of Step 1.

Step 2. Eliminate sporadic simple groups and the Tits group. By way of contradiction, we assume that S is a simple sporadic group or the Tits group and that $\mathrm{cd}(G) \subseteq \mathrm{cd}(S_n)$ with $n \geq 5$. The character degrees of S_n, where $5 \leq n \leq 31$ can be found in [6]. Moreover the character degrees of sporadic simple groups and the Tits group together with their automorphism groups are also available in [6]. It is routine to check that $\mathrm{cd}(G) \nsubseteq \mathrm{cd}(S_n)$ for any $5 \leq n \leq 31$ and any almost simple group G with socle S, where S is a sporadic simple group or the Tits group. Thus we can assume that $n \geq 32$. It follows from Lemma 2.1 that $d_2(S_n) = n(n-3)/2 \geq d_2(S_{32}) = 464$. By Lemma 2.7($i$) and Table 1, we only need to consider the following cases: $S \in \{O'N, HN, Ly, Th, Fi_{23}, J_4, Fi_{24}', B, M\}$.

(1) $S = O'N$. In this case, we have $|\mathrm{Out}(S)| = 2$ so that either $G = S$ or $G = S.2$. Assume first that $G = S = O'N$. Then $d_9(O'N) = 58653$ and since $n \geq 32$, by Lemma 2.1, $d_9(S_n) \geq 62496 > d_9(O'N)$, which contradicts Lemma 2.7(i). Now assume $G = O'N.2$. Then $d_7(G) = 58653 < 62496 \leq d_9(S_n)$ so that $d_7(G) \in \{d_7(S_n), d_8(S_n)\}$. However we can check that these equations cannot hold for any $n \geq 32$. Thus $\mathrm{cd}(G) \nsubseteq \mathrm{cd}(S_n)$.

(2) $S = HN$. Then $|\mathrm{Out}(S)| = 2$ so that $G = S$ or $G = S.2$. From [5], we have $d_7(S) = 16929$ and $d_7(S.2) = 17556$. Observe that $d_7(G) < 31000 \leq d_7(S_n)$ so that $\mathrm{cd}(G) \nsubseteq \mathrm{cd}(S_n)$ by Lemma 2.7(i).

(3) $S = Ly$. Since $|\mathrm{Out}(S)| = 1$, we have $G = S$ so that $p(S) = 67 \in \pi(S_n)$ by Lemma 2.7(ii), and hence $n \geq 67$. As $d_5(Ly) = 381766 < 718575 \leq d_7(S_n)$, we deduce that $d_5(Ly) \in \{d_5(S_n), d_6(S_n)\}$. However we can check that these equations cannot hold for any $n \geq 67$. Thus $\mathrm{cd}(G) \nsubseteq \mathrm{cd}(S_n)$.

(4) $S = Th$. As $\mathrm{Out}(S)$ is trivial, we have $G = S$. Since $d_1(G) = 248 < 464 \leq d_2(S_n)$, it follows from Lemma 2.7(i) that $d_1(G) = d_1(S_n) = n - 1$ and hence $n = 249$. But then $d_2(S_n) = n(n-3)/2 \geq 30627 > d_2(Th)$. Thus $\mathrm{cd}(G) \nsubseteq \mathrm{cd}(S_n)$.

(5) $S = Fi_{23}$. As $\mathrm{Out}(S)$ is trivial, we have $G = S$. Since $d_2(G) = 3588 < 4464 \leq d_4(S_n)$, it follows from Lemma 2.7(i) that $d_2(G) \in \{d_2(S_n), d_3(S_n)\}$. However we can check that these equations cannot hold for any $n \geq 32$. Thus $\mathrm{cd}(G) \nsubseteq \mathrm{cd}(S_n)$.

(6) $S = J_4$. Since $|\mathrm{Out}(S)| = 1$, we have $G = S$ so that $p(S) = 43 \in \pi(S_n)$ and hence $n \geq 43$. As $d_1(J_4) = 1333 < 11438 \leq d_4(S_n)$, we deduce that $d_1(J_4) \in \{d_i(S_n) \mid i = 1, 2, 3\}$. Solving these equations, we obtain $n = 1334$. But then $d_2(S_n) = 887777 > 299367 = d_2(J_4)$. Thus $\mathrm{cd}(G) \nsubseteq \mathrm{cd}(S_n)$.

(7) $S = Fi_{24}'$. We have $G = S$ or $G = S.2$. In both cases, we have $d_1(G) = 8671$ and $d_2(G) = 57477$. Observe that $d_1(G) = 8671 < 8960 \leq d_6(S_n)$ so that $d_1(G) \in \{d_i(S_n) \mid i = 1, \cdots, 5\}$. Solving these equations, we obtain $n = 8672$. But then $d_2(S_n) > d_2(G)$. Thus $\mathrm{cd}(G) \nsubseteq \mathrm{cd}(S_n)$.

(8) $S = B$. Since $|\mathrm{Out}(S)| = 1$, we have $G = S$ so that $p(S) = 47 \in \pi(S_n)$ and hence $n \geq 47$. As $d_1(B) = 4371 < 15134 \leq d_4(S_n)$, we deduce that $d_1(B) \in \{d_1(S_n), d_2(S_n), d_3(S_n)\}$. Solving these equations, we have $n = 95$ or $n = 4372$. If the latter case holds then $d_2(S_n) > d_2(B)$, a contradiction. Thus $n = 95$. But then $d_3(S_{95}) = 4371 < d_2(B) < d_4(S_{95}) = 133950$. Thus $\mathrm{cd}(G) \nsubseteq \mathrm{cd}(S_n)$.

(9) $S = M$. Since $|\mathrm{Out}(S)| = 1$, we have $G = S$ so that $p(S) = 71 \in \pi(S_n)$ and hence $n \geq 71$. As $d_1(M) = 196883 < 914480 \leq d_7(S_n)$, we deduce that $d_1(M) \in \{d_i(S_n) \mid i = 1, \cdots, 6\}$. Solving these equations, we obtain $n = 196884$. But then $d_2(S_n) > 21296876 = d_2(M)$. Thus $\mathrm{cd}(M) \nsubseteq \mathrm{cd}(S_n)$.

Step 3. If $S \cong A_m$, with $m \geq 5$, then $m = n$. Let $\lambda = (m-1, 1)$ be a partition of m. Since $m \geq 5$, λ is not self-conjugate so that the irreducible character χ^λ of S_m corresponding to λ is still irreducible upon restriction to A_m. Note that $\mathrm{Aut}(A_m) = S_m$ whenever $m \neq 6$ while $|\mathrm{Out}(A_6)| = 4$. Assume first that $m \neq 6$. Then $G \in \{A_m, S_m\}$ and G contains an irreducible character of degree $m-1$. Since $\mathrm{cd}(G) \subseteq \mathrm{cd}(S_n)$, we have $m - 1 \geq d_1(S_n) = n - 1$ so that $m \geq n$. If $m = n$ then we are done. Hence we assume that $m > n \geq 5$. It follows that $m \geq 6$ and hence $d_1(A_m) = d_1(S_m) = m - 1$ and thus $d_1(G) = m - 1 > n - 1 = d_1(S_n)$. If $m \leq 17$ then $5 \leq n < m \leq 17$. Using [6], we can check that $m = n$. So we can assume that $m \geq 18$. It follows that $17 \in \pi(G)$ and so by Lemma 2.7(ii) we have $17 \in \pi(S_n)$, which implies that $n \geq 17$. Thus $17 \leq n < m$. It follows from Lemma 2.1 that $d_2(S_n) = n(n-3)/2$. Since $\mathrm{cd}(G) \subseteq \mathrm{cd}(S_n)$ and $d_1(G) > d_1(S_n)$, it follows that $d_1(G) \geq d_2(S_n)$. Then $m - 1 \geq n(n-3)/2$. Since $n \geq 17$, we have $m - 2n \geq n(n-3)/2 + 1 - 2n = n(n-7)/2 + 1 > 0$ so that $m > 2n$. Therefore $n < m/2 < m$. By Lemma 2.2, there exists a prime p such that $m/2 \leq p < m$. It follows that $p \in \pi(G)$ but $p \notin \pi(S_n)$ since $p > n$, which contradicts Lemma 2.7(ii). Thus $S \cong A_n$ whenever $m \geq 5, m \neq 6$. Now assume that $m = 6$, and $A_6 \trianglelefteq G \leq \mathrm{Aut}(A_6)$. It follows that $G \in \{A_6, A_6.2_1 \cong S_6, A_6.2_2 \cong PGL_2(9), A_6.2_3 \cong M_{10}, A_6.2^2\}$. We need to show that $n = 6$. If $G \in \{A_6, S_6\}$, then G contains a character of degree 5 so that $5 \geq n - 1$ and hence $n \leq 6$. As $10 \in \mathrm{cd}(G)$ but $10 \notin \mathrm{cd}(S_5)$, we conclude that $n = 6$. If $G \cong PGL_2(9)$ then $\{8, 9\} \subseteq \mathrm{cd}(S_n)$. But this cannot happen by Lemma 2.3. Assume that one of the last two cases holds. Then $\{9, 16\} \subseteq \mathrm{cd}(S_n)$ so that by Lemma 2.3, $n = 6$. The proof is now complete. □

Remark 3.2. *Let* $\lambda = (k+1, 1^k)$ *when* $n = 2k+1$ *and* $\lambda = (k, 2, 1^{k-2})$ *when* $n = 2k$. *Then* λ *is a self-conjugate partition of* n. *We conjecture that* $\chi^\lambda(1)/2 \in \mathrm{cd}(A_n) - \mathrm{cd}(S_n)$ *and* $\chi^\lambda(1) \in \mathrm{cd}(S_n) - \mathrm{cd}(A_n)$. *If this conjecture is true then, in the situation of Theorem 3.1, we deduce that* $G \cong S_n$ *or* $G \cong M_{10}$ *and* $n = 6$. *This result will be useful in studying Huppert's Conjecture for alternating groups.*

Theorem 3.3. *Let G be a group. Assume that $|G : G'| = 2$ and that $G' \cong S^2$ is a unique minimal normal subgroup of G, where S is a non-abelian simple group. Then $\mathrm{cd}(G) \nsubseteq \mathrm{cd}(S_n)$ for any $n \geq 5$.*

Proof. By way of contradiction, assume that $\mathrm{cd}(G) \subseteq \mathrm{cd}(S_n)$. Let $\alpha \in \mathrm{Irr}(S)$ with $\alpha(1) > 1$ and put $\theta = \alpha \times 1 \in \mathrm{Irr}(G')$. Observe that θ is not G-invariant so that $I_G(\theta) = G'$ hence $\theta^G \in \mathrm{Irr}(G)$ and so $\theta^G(1) = 2\alpha(1) \in \mathrm{cd}(S_n)$. On the other hand, if $\varphi = \alpha \times \alpha \in \mathrm{Irr}(G')$ then φ is G-invariant and since G/G' is cyclic, we deduce that φ extends to $\psi \in \mathrm{Irr}(G)$, so that $\psi(1) = \alpha(1)^2 \in \mathrm{cd}(S_n)$. We conclude that if $a \in \mathrm{cd}(S) - \{1\}$ then $2a, a^2 \in \mathrm{cd}(S_n)$. Let $r \in \pi(S)$. By Corollary 2.6, $r|a$ for some $a \in \mathrm{cd}(S) - \{1\}$. Since $a^2 \in \mathrm{cd}(S_n)$, by [8, Theorem 3.11] we have $a^2|n!$. Thus $r^2|n!$ so that $n \geq 2r$. Using the classification of finite simple groups, we consider the following cases.

Case $S = A_m$, with $m \geq 5$. As $m \geq 5$, it follows from the first paragraph that $n \geq 10$. Assume first that $m \in \{5, 6, 8, 9, 10\}$. Observe that $m - 1 \in \mathrm{cd}(S)$ and hence $2(m-1), (m-1)^2 \in \mathrm{cd}(S_n)$. For these values of m, we see that $(m-1)^2$ is a prime power and so by Lemma 2.3, as $n \geq 10$, we have $d_1(S_n) = n - 1 = (m-1)^2$. As $m \geq 5$, we obtain $d_1(S_n) = (m-1)^2 > 2(m-1) > 1$, which is a contradiction since $2(m-1) \in \mathrm{cd}(S_n)$. Now assume that $m = 7$. As above, we have $n \geq 14$. As $6 \in \mathrm{cd}(S)$, we obtain $6.2 = 12 \in \mathrm{cd}(S_n)$ and so $12 \geq d_1(S_n) = n - 1$ which implies $n \leq 13$, a contradiction. Thus we can assume that $m \geq 11$ and hence $n \geq 22$. We have $m - 1 \in \mathrm{cd}(S)$ so that $2(m-1)$ and $(m-1)^2$ are both in $\mathrm{cd}(S_n)$. Similarly, by Lemma 2.1, we have $m(m-3)/2, (m-1)(m-2)/2 \in \mathrm{cd}(S)$ and so $m(m-3), (m-1)(m-2) \in \mathrm{cd}(S_n)$. We will show that $m < n$. By way of contradiction, assume that $m \geq n$. As $n \geq 22$, by Lemma 2.2, there exists a prime r such that $n/2 < r \leq n$. It follows that the largest power of r dividing $n!$ is r. Since $r \leq n \leq m$, we deduce that $r \in \pi(A_m)$ and so $r^2|n!$, which is a contradiction. Thus $m < n$. Observer that $1 < 2(m-1) < m(m-3) < (m-1)(m-2) < (m-1)^2$, since $m \geq 11$. Thus $(m-1)^2 \geq d_4(S_n) = n(n-1)(n-5)/6$ by Lemma 2.1. Combining with the fact that $m < n$, we obtain $n(n-1)(n-5)/6 \leq (n-1)^2$ so that $n(n-5) \leq 6(n-1)$ or equivalently $n(n-11) + 6 \leq 0$, which is impossible as $n \geq 22$.

Case S is a finite simple group of Lie type in characteristic p, with $S \neq {}^2F_4(2)'$. Since $L_2(4) \cong L_2(5) \cong A_5$, we can assume that $S \ncong L_2(4)$. Let St be the Steinberg character of S. We can check that $St(1) = |S|_p \geq 5$. Since $St(1) \in \mathrm{cd}(S)$, we obtain $2St(1) \in \mathrm{cd}(S_n)$ and $St(1)^2 \in \mathrm{cd}(S_n)$. By Lemma 2.3, assume first that $n - 1 = St(1)^2$. Then $d_1(S_n) = St(1)^2 > 2St(1)$, which is a contradiction. Now assume that $St(1)^2 \neq n - 1$. By Lemma 2.3, $n \in \{5, 6, 8, 9\}$. Since $St(1)^2$ is an even prime power, one of the following cases holds: $n = 6$, $St(1)^2 \in \{3^2, 2^4\}$ or $n = 8, St(1)^2 = 2^6$. Assume first that $n = 6$. Then $St(1) \in \{3, 2^2\}$ which implies that $St(1) \leq 4$, a contradiction as $S \neq L_2(4)$. Finally, assume that $n = 8$. Then $St(1) = 2^3 = 8$. However $2St(1) = 2^4 \notin \mathrm{cd}(S_8)$, a contradiction.

Case S is a sporadic simple group or the Tits group. Recall that $p(S)$ is the largest prime divisor of S. We have $n \geq 2p(S)$.

(1) $S \in \{M_{11}, M_{12}, J_1, M_{22}, J_2, M_{23}, HS, J_3, M_{24}, He, Ru, Co_3, Co_2, Co_1, {}^2F_4(2)'\}$. These cases can be eliminated as follows: Since $n \geq 2p(S) \geq 14$, we have $d_2(S_n) = n(n-3)/2 \geq p(S)(2p(S) - 3)$. Next observe that $2d_i(S) \in \mathrm{cd}(G) \subseteq \mathrm{cd}(S_n)$, for $i = 1, 2$ and that $1 < 2d_1(S) < 2d_2(S)$. In each case, we have $p(S)(2p(S) - 3) > 2d_2(S)$ and hence $d_2(S_n) > 2d_2(S) > 2d_1(S) > 1$, which contradicts Lemma 2.7(i).

(2) $S \in \{McL, Suz, Fi_{22}, HN, Ly, Th, J_4, B\}$. Since $n \geq 2p(S) \geq 14$, we have $d_2(S_n) = n(n-3)/2 \geq p(S)(2p(S)-3)$. We have $d_2(S_n) \geq p(S)(2p(S)-3) > 2d_1(S)$ and so $2d_1(S) = d_1(S_n) = n-1$ so that $n = 2d_1(S)+1$. But then $d_2(S_n) = n(n-3)/2 = (d_1(S)-1)(2d_1(S)+1) > 2d_2(S) > 2d_1(S) > 1$, which contradicts Lemma 2.7(i) as $2d_i(S) \in \mathrm{cd}(G) \subseteq \mathrm{cd}(S_n)$, where $i = 1, 2$.
(3) $S = O'N$. Then $p(S) = 31$. We have $n \geq 2p(S) = 62$ and so by Lemma 2.1, $d_7(S_n) \geq 520025$. As $d_8(S) = 58311$, we have $2d_8(S) = 116622 \in \mathrm{cd}(S_n)$. Note that $2d_i(S) \in \mathrm{cd}(S_n)$ for $i = 1, 2, \cdots, 8$ and so $2d_i(S) \geq d_i(S_n)$ for all $1 \leq i \leq 8$. As $d_7(S_n) \geq 520025 > 116622 = 2d_8(S)$, we get a contradiction.
(4) If $S = Fi_{23}$ then $p(S) = 23$. We have $n \geq 2p(S) = 46$ and so by Lemma 2.1, $d_4(S_n) \geq 14145$. As $d_2(S) = 3588$, we obtain $2d_2(S) = 7176 \in \mathrm{cd}(S_n)$. Since $d_4(S_n) > 7176 > 2d_1(S)$, we must have $7176 \in \{d_2(S_n), d_3(S_n)\}$. However, we can check that these cases cannot happen.
(5) $S = Fi'_{24}$. Then $p(S) = 29$ and $n \geq 2p(S) = 58$ so that by Lemma 2.1, $d_4(S_n) \geq 29203$. As $\{8671, 57477\} \subseteq \mathrm{cd}(S)$, we obtain $\{17342, 114954\} \subseteq \mathrm{cd}(S_n)$. Since $d_4(S_n) > 17342$, we have $17342 \in \{d_1(S_n), d_2(S_n), d_3(S_n)\}$. It follows that $17342 \leq (n-1)(n-2)/2$ and hence $n \geq 188$. But then $d_2(S_n) \geq 17390 > 17342$. Thus $d_1(S_n) = n-1 = 17342$ hence $n = 17343$ and so $d_2(S_n) \geq 150363810 > 114954$, a contradiction.
(6) $S = M$. Then $p(S) = 71$ and $n \geq 2p(S) = 142$. By Lemma 2.1, $d_4(S_n) \geq 457169$. As $\{196883, 21296876\} \subseteq \mathrm{cd}(S)$, we obtain $\{393766, 42593752\} \subseteq \mathrm{cd}(S_n)$. Since $d_4(S_n) > 393766$, we have $393766 \in \{d_1(S_n), d_2(S_n), d_3(S_n)\}$. It follows that $393766 \leq (n-1)(n-2)/2$ and hence $n \geq 889$. As $d_2(S_n) \geq 393827 > 393766$, we have $d_1(S_n) = n-1 = 393766$ hence $n = 393767$ and so $d_2(S_n) \geq 77525634494 > 42593752 > 393766$, a contradiction. The proof is complete. □

Proof of Theorem 1.1. Suppose that $X_1(G) = X_1(S_n)$. It follows that $|G| = |S_n| = n!$, $|G : G'| = 2$, $k(G) = k(S_n)$ and $\mathrm{cd}(G) = \mathrm{cd}(S_n)$. For $n \leq 3$, the result is trivial. If $n = 4$, then the result follows from [2, Chapter 17, Exercise 2]. Thus from now on, we assume that $n \geq 5$.

We first show that $G' = G''$. By way of contradiction, assume that $G'' < G'$. Let $N \leq G'$ be a normal subgroup of G maximal such that G/N is solvable and G'/N is the unique minimal normal subgroup of G/N. By [8, Lemma 12.3], all non-linear irreducible characters of G/N have equal degree f and either G/N is a p-group, $Z(G/N)$ is cyclic and $G/N/Z(G/N)$ is elementary abelian of order f^2 or G/N is a Frobenius group with an abelian Frobenius complement of order f, and G'/N is the Frobenius kernel and is an elementary abelian p-group. Assume first that G/N is a p-group. As $G/N/Z(G/N)$ is abelian, we have $G'/N \leq Z(G/N)$. Since $|G/N : G'/N| = 2$ and G/N is non-abelian, we deduce that $G'/N = Z(G/N)$ and so $G/N/Z(G/N)$ is a cyclic group of order 2, which is a contradiction as $G/N/Z(G/N)$ is elementary abelian of order f^2. Thus the second situation holds. It follows that $f = |G/N : G'/N| = |G : G'| = 2$. Therefore $2 \in \mathrm{cd}(G) = \mathrm{cd}(S_n)$, which is impossible as the minimal non-trivial irreducible character degree of S_n is $n-1 \geq 4$ as $n \geq 5$.

Let $M \leq G'$ be a normal subgroup of G so that G'/M is a chief factor of G and so $G'/M \cong S^k$, where S is a non-abelian simple group where $k \geq 1$. Let $C/M = C_{G/M}(G'/M)$. Then $M \leq C \trianglelefteq G$.

Assume first that $C = M$. Then G'/M is the unique minimal normal subgroup of G/M. Since $|G/M : G'/M| = |G : G'| = 2$, we deduce that k is at most 2. However

k cannot be 2 by Theorem 3.3. Thus $k = 1$ and so G/M is an almost simple group with socle G'/M and $\mathrm{cd}(G/M) \subseteq \mathrm{cd}(S_n)$. By Theorem 3.1, we have $G'/M \cong A_n$. It follows that $|G/M| = 2|G'/M| = n! = |S_n|$ and so $M = 1$ as $|G| = |S_n|$. Thus G is an almost simple group with socle A_n and $|G| = n!$. If $n \neq 6$ then as $\mathrm{Aut}(A_n) = S_n$ we deduce that $G \cong S_n$. Now assume that $n = 6$. Then G is isomorphic to one of the following groups $S_6 \cong A_6.2_1, PGL_2(9) \cong A_6.2_2$ and $M_{10} \cong A_6.2_3$. Using [5], we can see that G must be isomorphic to S_6.

Finally assume that $C \neq M$. It follows that C/M is a non-trivial normal subgroup of G/M and so $C/M \cap G'/M$ is trivial. Thus $G' < G'C \leq G$. Since $|G : G'| = 2$, we deduce that $G = G'C$ and hence $G/M = G'/M \times C/M$, where C/M is a cyclic subgroup of order 2. Thus $\mathrm{cd}(G'/M) = \mathrm{cd}(G/M) \subseteq \mathrm{cd}(S_n)$. Applying Theorem 3.1 again, we obtain $G'/M \cong A_n$, and hence $G/M \cong A_n \times \mathbb{Z}_2$. By comparing the orders, we deduce as in previous case that $M = 1$ and so $G \cong A_n \times \mathbb{Z}_2$. We now show that this case cannot happen. In fact, we have $k(G) = 2k(A_n)$, and hence it suffices to show that $k(S_n) < 2k(A_n)$. Let λ be a partition of n and denote by χ^λ the irreducible character of S_n associated to λ. If λ is not self-conjugate, that is $\lambda \neq \lambda'$, where λ' denotes the conjugate of λ, then $(\chi^\lambda)_{A_n} = (\chi^{\lambda'})_{A_n} \in \mathrm{Irr}(A_n)$. Otherwise, $(\chi^\lambda)_{A_n} = \chi^{\lambda+} + \chi^{\lambda-}$, where $\chi^{\lambda+}, \chi^{\lambda-} \in \mathrm{Irr}(A_n)$ are two non-equivalent irreducible characters of the same degree. Let $p_s(n)$ be the number of self-conjugate partitions of n. Then $k(A_n) = (k(S_n) - p_s(n))/2 + 2p_s(n)$. Hence $k(S_n) = 2k(A_n) - 3p_s(n)$. So it suffices to show that $p_s(n) \geq 1$ for $n \geq 5$. In fact, if $n = 2l + 1$, then we take $\lambda = (l + 1, 1^l)$ and if $n = 2l$ then take $\lambda = (l, 2, 1^{l-1})$. Then λ is a self-conjugate partition of n so that $p_s(n) \geq 1$. This finishes the proof. □

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