p-PARTS OF BRAUER CHARACTER DEGREES

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ABSTRACT. Let G be a finite group and let p be an odd prime. Under certain conditions on the p-parts of the degrees of its irreducible p-Brauer characters, we prove the solvability of G. As a consequence, we answer a question proposed by B. Huppert in 1991: If G has exactly two distinct irreducible p-Brauer character degrees, then G is solvable. We also determine the structure of non-solvable groups with exactly two irreducible 2-Brauer character degrees.

1. INTRODUCTION

In the late 1980's and earlier 90's there was considerable interest in studying the degrees of the irreducible p-Brauer characters of finite p-solvable groups. For general finite groups, however, not much was proven; partly because the knowledge of modular representation theory was not as it is nowadays, and partly due to the somehow erratic behavior of the degrees of the representations in characteristic p.

For instance, an apparently innocent question by B. Huppert was left unsolved: If all the non-linear irreducible p-Brauer characters have the same degree, where p is odd, is G solvable? Huppert was of course aware that for $p = 2$, the irreducible 2-Brauer degrees of $\mathrm{PGL}_2(q)$ are 1 and $q - 1$, whenever $q = 9$ or a Fermat prime, so this was really a question for odd primes.

Although this paper started as an attempt to settle Huppert's question, we soon realized that the key in Huppert's problem relied on the p-parts of the p-Brauer character degrees. To start our discussion, suppose that we have fixed a maximal ideal of the ring of algebraic integers containing the prime p, with respect to which we calculate the irreducible p-Brauer characters $\varphi \in \mathrm{IBr}_p(G)$ of every finite group G. Let us denote $\mathrm{cd}_p(G) = \{\varphi(1) \,|\, \varphi \in \mathrm{IBr}_p(G)\}$.

In general, not much can be said about the degrees of p-Brauer characters of arbitrary finite groups. However, p-Brauer character degrees display a slightly better

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Date: February 4, 2014.

2000 *Mathematics Subject Classification.* Primary 20C20; Secondary 20D05, 20D10.

Key words and phrases. Brauer characters; solvable groups.

The authors are grateful to the referee for careful reading and helpful comments on the paper.

The research of the first author is supported by the Prometeo/Generalitat Valenciana, Proyectos MTM2010-15296. The second author gratefully acknowledges the support of the NSF (grants DMS-0901241 and DMS-1201374). The research of the third author is supported by a Startup Research Fund from the College of Agriculture, Engineering and Science, the University of KwaZulu-Natal.

behavior if we consider their p-parts. For instance, a theorem of G. Michler [M] asserts that $\varphi(1)_p = 1$ for all $\varphi \in \mathrm{IBr}_p(G)$ (that is, p does not divide $\varphi(1)$ for all $\varphi \in \mathrm{IBr}_p(G)$) if and only if G has a normal Sylow p-subgroup. Since $\mathrm{cd}_p(G) = \mathrm{cd}_p(G/\mathbf{O}_p(G))$ (because $\mathbf{O}_p(G)$ is in the kernel of the irreducible p-modular representations), we see that when dealing with p-Brauer character degrees, we may generally assume that p divides some $m \in \mathrm{cd}_p(G)$.

Theorem A. *Let G be a finite group and let p be an odd prime. Suppose that the degrees of all nonlinear irreducible p-Brauer characters of G are divisible by p.*

(i) *If $p \geq 5$ then G is solvable.*

(ii) *If $p = 3$ and the p-parts of the degrees of non-linear irreducible p-Brauer characters of G take at most two different values, then G is solvable.*

We note that if $G = \mathrm{PSL}_2(27) \cdot 3$, then we have that $\mathrm{cd}_3(G) = \{1, 9, 12, 27, 36\}$, which shows that Theorem A(i) fails for $p = 3$ and that Theorem A(ii) is best possible. We should also mention that under the conditions of Theorem A, the prime 2 behaves somehow in the opposite way: it is often the case that all non-linear irreducible 2-Brauer characters of non-solvable groups have even degree; in fact, the number of their 2-parts can be quite large (with the exception of M_{22} where all non-linear 2-Brauer character degrees have the same 2-part).

As a consequence of Theorem A, we can answer B. Huppert's question mentioned before ([H, p. 27]).

Theorem B. *Let G be a finite group and let p be an odd prime. Suppose that $\mathrm{cd}_p(G) = \{1, m\}$ with $m > 1$. Then G is solvable.*

Since $\mathrm{cd}_5(\mathsf{A}_5) = \{1, 3, 5\}$, we see that Theorem B cannot be further generalized.

As we have already said, $\mathrm{cd}_2(\mathrm{PGL}_2(q)) = \{1, q-1\}$ whenever $q = 9$ or a Fermat prime, showing that for $p = 2$, there are non-solvable groups satisfying the hypothesis of Theorem B. But in fact, we have the following.

Theorem C. *Let G be a non-solvable group with $\mathbf{O}_2(G) = 1$. Then $\mathrm{cd}_2(G) = \{1, m\}$ with $m > 1$ if and only if the following conditions hold:*

(i) *$m = 2^a$ for some $a \geq 2$, $q := 2^a + 1$ is either a Fermat prime or $q = 9$; and*

(ii) *G has a normal subgroup $S \cong \mathrm{PSL}_2(q)$, $G/(S \times \mathbf{Z}(G)) \cong C_2$, and G induces the group of inner-diagonal automorphisms of S.*

Finally, we mention that groups whose irreducible p-Brauer characters have prime power degrees have recently received attention in [TW] and [TV].

2. Proof of Theorem A

Let G be a finite group and let p be a prime. We follow the notation in [I2] for complex characters and in [N1] for p-Brauer characters. We write $\mathrm{Irr}(G)$ for the set of all complex irreducible characters of G, and $\mathrm{cd}(G) = \{\chi(1) \,|\, \chi \in \mathrm{Irr}(G)\}$ for the set of their degrees. Similarly, $\mathrm{IBr}_p(G)$ is the set of all irreducible p-Brauer characters of G (when a maximal ideal of the algebraic integers containing p has been fixed) and $\mathrm{cd}_p(G) = \{\varphi(1) \mid \varphi \in \mathrm{IBr}_p(G)\}$. For an integer $n \geq 1$, we denote the largest power of p which divides n by n_p, the p-part of n. If $N \trianglelefteq G$ and $\lambda \in \mathrm{IBr}_p(N)$ then $\mathrm{IBr}_p(G|\lambda)$ is the set of all constituents of the induced character λ^G in $\mathrm{IBr}_p(G)$. If $\chi \in \mathrm{Irr}(G)$, then χ° is the restriction of χ to G°, the set of all p-regular elements of G. The set of all prime divisors of the order of G is denoted by $\pi(G)$. Other notation is standard.

Proof of Theorem A. Let G be a counterexample with minimal order. We first observe that if $N \trianglelefteq G$, then $\mathrm{cd}_p(G/N) \subseteq \{1\} \cup p\mathbb{Z}$ since $\mathrm{cd}_p(G/N) \subseteq \mathrm{cd}_p(G)$.

Step 1. $G = \mathbf{O}^{p'}(G)$.

Let $H = \mathbf{O}^{p'}(G)$. Then $H \trianglelefteq G$ and G/H is a p'-group. It follows that $\mathrm{IBr}_p(G/H) = \mathrm{Irr}(G/H)$ and so $\mathrm{cd}_p(G/H) = \mathrm{cd}(G/H)$. Since $\mathrm{cd}_p(G/H) \subseteq \{1\} \cup p\mathbb{Z}$, we see that the degree of every non-linear (if any) complex irreducible character of G/H is divisible by p, which forces $\mathrm{cd}(G/H) = \{1\}$. In particular, G/H is an abelian p'-group. Note that H also satisfies the same hypothesis that we have on G. Indeed, let $\theta \in \mathrm{IBr}_p(H)$ with $\theta(1) > 1$ and let $\varphi \in \mathrm{IBr}_p(G|\theta)$. As G/H is solvable, we have that $\varphi(1)/\theta(1)$ divides $|G/H|$ by [N1, Theorem 8.22] and thus $\theta(1)_p = \varphi(1)_p$ since $|G/H|$ is coprime to p. Now the minimality of $|G|$ implies that $G = H$.

Step 2. Every Sylow p-subgroup of G is self-normalizing.

As the number of linear p-Brauer characters of G is $|G : G'\mathbf{O}^{p'}(G)|$, by Claim 1 we deduce that the trivial Brauer character 1_G is the only linear p-Brauer character of G. It follows that $p \mid \beta(1)$ for all $1_G \neq \beta \in \mathrm{IBr}_p(G)$. The result now follows from [NT1, Theorem A].

Step 3. $p = 3$ and G has a composition factor which is isomorphic to $\mathrm{PSL}_2(3^f)$ with $f = 3^a \geq 3$. In particular, we are done with the proof of Theorem A(i).

By way of contradiction, suppose that either $p \geq 5$ or $p = 3$ and G has no composition factor which is isomorphic to $\mathrm{PSL}_2(3^f)$ for some $f = 3^a$ and $a \geq 1$. Then G satisfies the hypothesis of Theorem 1.1 in [GMN] and thus G is solvable, which is a contradiction.

Step 4. Set $p = 3$. Now we reach the final contradiction by showing that $\mathrm{IBr}_3(G)$ contains three characters α, β, and γ with

$$\alpha(1)_3 > \beta(1)_3 > \gamma(1)_3,\ \gamma(1) > 1.$$

By Step 3, we may assume that G has a chief factor $N/K = S_1 \times \ldots \times S_n$ with $S_i \cong S = \mathrm{PSL}_2(3^f)$ and $f = 3^a \geq 3$. We will produce three irreducible 3-Brauer characters of G/K with desired properties. Hence without loss we may assume that

$K = 1$ and $N = S_1 \times \cdots \times S_n$ is a minimal normal subgroup of G. Firstly, let St denote the Steinberg character of S and consider the character $\mathsf{St} \times \ldots \times \mathsf{St}$ of degree 3^{fn} of N. By the main result of [F], this character extends to G and is irreducible modulo 3. Thus we obtain $\alpha \in \mathrm{IBr}_3(G)$ with $\alpha(1) = 3^{fn}$.

Let $V = \overline{\mathbb{F}}_3^2$ denote the natural module for $\mathrm{SL}_2(3^f)$. Then

$$\mathrm{Sym}^2(V) \otimes \mathrm{Sym}^2(V)^{(3)} \otimes \mathrm{Sym}^2(V)^{(3^2)} \otimes \cdots \otimes \mathrm{Sym}^2(V)^{(3^{(f-3)/2})}$$

is an irreducible S-module of dimension $3^{(f-1)/2}$, with character say β_1. Consider the Brauer character

$$\beta_0 = \beta_1 \times 1_{S_2} \times \cdots \times 1_{S_n}$$

of N. Then the inertia group $I_G(\beta_0)$ is contained in $M := \mathbf{N}_G(S_1)$ and contains CS_1, with $C := \mathbf{C}_G(S_1)$. Note that $M/CS_1 \leq \mathrm{Out}(S_1) \cong C_{2f}$ is cyclic (since f is odd). Hence, working in $I_G(\beta_0)/C$, we see that β_0 extends to $I_G(\beta_0)$. Now the Clifford correspondence yields $\beta \in \mathrm{IBr}_3(G|\beta_0)$ with $\beta(1) = 3^{(f-1)/2}bn$, where $b := |M : I_G(\beta_0)|$ divides $2f$.

Next, $V \otimes V^{(3)}$ is also an irreducible S-module of dimension 4, with character say γ_1. Considering the Brauer character

$$\gamma_0 = \gamma_1 \times 1_{S_2} \times \cdots \times 1_{S_n}$$

of N and arguing as above, we obtain $\gamma \in \mathrm{IBr}_3(G|\gamma_0)$, where $\gamma(1) = 4cn > 1$ and $c := |M : I_G(\gamma_0)|$ divides $2f$.

Suppose for the moment that $a \geq 2$ and so $f \geq 9$. Then $b_3, c_3 \leq f < 3^{(f-3)/2}$, whence

$$\gamma(1)_3 = (4cn)_3 < n_3 \cdot 3^{(f-3)/2} < \beta(1)_3 = (3^{(f-1)/2}bn)_3 < 3^{f-2}n < 3^{fn} = \alpha(1)_3,$$

and so we are done.

Finally, assume that $f = 3$. Note that β_1 and γ_1 are both fixed by the diagonal automorphism of S_1, but none of them is fixed by the field automorphism (of order 3) of S_1. It follows that they have the same inertia group in M/C, whence $I_G(\beta_0) = I_G(\gamma_0)$ and $b = c$. Recall also that $b|2f = 6$. Hence,

$$\gamma(1)_3 = (4cn)_3 = (bn)_3 < \beta(1)_3 = (3bn)_3 \leq 9n < 27^n = \alpha(1)_3,$$

and we are again done.

3. Proof of Theorem B

Now we are ready to establish Theorem B.

Proof of Theorem B. Assume first that p does not divide m. By [M, Theorem 5.5], G has a normal Sylow p-subgroup P. Then $\mathrm{cd}_p(G) = \mathrm{cd}_p(G/P) = \mathrm{cd}(G/P)$ since G/P is a p'-group. It follows that $|\mathrm{cd}(G/P)| = 2$ and thus G/P is solvable by [I2, Corollary 12.5], hence G is solvable. Assume now that $p \mid m$ and so $m_p = p^a$

with $a \geq 1$. It follows that G is solvable by applying Theorem A. The proof is now complete. □

Solvable groups with two Brauer degrees were studied by F. Bernhardt in the earlier 90's. In B. Huppert's survey [H], a result by Bernhardt concerning their derived length is mentioned (although it seems that this result has never appeared in print). In any case, we take this opportunity to prove something that perhaps has not been noticed until now.

Theorem 3.1. *Let G be a finite group and let p be an odd prime. Suppose that $\mathrm{cd}_p(G) = \{1, m\}$ with $m > 1$ and $\mathbf{O}_p(G) = 1$. If $P \in \mathrm{Syl}_p(G)$, then P is cyclic.*

Recall that if $\varphi \in \mathrm{IBr}_p(G)$, then $\ker(\varphi)$ means the kernel of any modular representation affording φ. We need the following elementary and well-known lemma.

Lemma 3.2. *We have that*

$$\mathbf{O}_p(G) = \bigcap_{\varphi \in \mathrm{IBr}_p(G)} \ker(\varphi)\,.$$

Proof. If L is the intersection on the right hand side above and $\tau \in \mathrm{IBr}_p(L)$, then τ lies under some $\gamma \in \mathrm{IBr}_p(G)$ by [N1, Corollary 8.7]. Then $\tau = 1_L$ and L is a p-group by [N1, Corollary 2.10]. Also, $\mathbf{O}_p(G)$ is contained in every kernel by [N1, Lemma 2.32]. □

Proof of Theorem 3.2. If p does not divide m, then G has a normal Sylow p-subgroup by Michler's theorem [M, Theorem 5.5], and by the hypothesis, we deduce that G is a p'-group. In this case, everything is clear. So we may assume that $p|m$. By Theorem B, we know that G is solvable. Suppose that $q \neq p$ is a prime dividing m. By the [N2, Corollary], we have that G has a normal q-complement. Hence, if π is the set of primes dividing $m_{p'}$, it follows that G has a normal π-complement K. Hence G/K is a π-group and K is a π'-group. Let $\tau \in \mathrm{IBr}_p(K)$ be non-linear. Since K is p-solvable, we have that $\tau(1)$ divides $|K|$ (for instance, use [N1, Theorem 10.1]). So does the determinantal order $o(\tau)$. (Recall that the determinantal order $o(\tau)$ is the order of the homomorphism $\Lambda : K \to \mathbb{F}^\times$ that affords the Brauer character $\det(\tau)$, where $\mathbb{F}$ is an algebraically closed field of characteristic p, in the group $\mathrm{Hom}(K, \mathbb{F}^\times)$.) Let T be the stabilizer of τ in G. By [N1, Theorem 8.23], we have that τ extends to some $\gamma \in \mathrm{IBr}_p(T)$. Also, $\gamma^G \in \mathrm{IBr}_p(G)$ by the Clifford correspondence. Thus $m = \tau(1)|G : T|$, and we deduce that $m_p = m_{\pi'} = \tau(1)$. Since G/K is a p'-group, then we may assume that $K = G$ and that $m = p^a > 1$ is a power of p.

Now, let $L = \mathbf{O}^p(G')$. We claim that L is abelian. By Lemma 3.2, it suffices to show that L' is contained in $\ker(\varphi)$ for every $\varphi \in \mathrm{IBr}_p(G)$. This is clear if $\varphi(1) = 1$, so suppose that $\varphi(1) = p^a$ with $a > 0$. Now let $\chi \in \mathrm{B}_p(G)$ be an Isaacs lifting of φ

(see [I1] for the definition of $\mathrm{B}_p(G)$). Since by definition χ is induced from a p'-degree character and $\chi(1)$ has p-power degree, then we conclude that χ is monomial. Hence, let $\lambda \in \mathrm{Irr}(H)$ be linear such that $\lambda^G = \chi$. Now, consider the Brauer character $(1_H)^G$. By Nakayama's Lemma [N1, Lemma 8.4], we have that the trivial Brauer character 1_G is a constituent of the Brauer character $(1_H)^G$. Since $|G : H| = p^a$, it follows that all the irreducible Brauer constituents of $(1_H)^G$ are necessarily linear. Thus $((1_H)^G)_{G'}$ is a multiple of the trivial Brauer character $1_{G'}$. By Mackey ([N1, Problem 8.5]), it follows that $(1_{H\cap G'})^{G'}$ is also a multiple of $1_{G'}$. Hence, if $z \in G'$ is p-regular, then we have that

$$(1_{H\cap G'})^{G'}(z) = (1_{H\cap G'})^{G'}(1)\,.$$

By the induction of Brauer characters formula, we deduce that $\mathrm{core}_{G'}(H \cap G')$ contains every p-regular element of G'. Hence $\mathbf{O}^p(G') \leq H$ and $\mathbf{O}^p(G')' \leq H'$. Then $\mathbf{O}^{p'}(G')' \leq \mathrm{core}_G(\ker(\lambda)) = \ker(\chi)$. This proves the claim. Since $\mathbf{O}_p(G) = 1$, we see that L is a p'-group. Now, if R/G' is the Sylow p-subgroup of G/G', we have that R/L is a normal Sylow p-subgroup of G/L, so we deduce that G has p-length one.

Now, notice that $M = \mathbf{O}_{p'}(G)$ is abelian, because all of its irreducible characters have p'-degree but lie below a p-power degree Brauer character. Let $N/M = \mathbf{O}_p(G/M)$. We have that $C = \mathbf{C}_M(P) < M$, because otherwise, $\mathbf{O}_p(G) > 1$. Also $\mathbf{C}_P(M) \leq \mathbf{O}_p(N) = 1$. Notice that P acts faithfully on M/C, because if $Q \leq P$ centralizes M/C, then $[M,Q] \leq C$, $[M,Q,Q] = 1$ and therefore $[M,Q] = 1$ by coprime action and $Q \leq \mathbf{C}_P(M) = 1$.

Suppose that $1 \neq \lambda \in \mathrm{Irr}(M/C)$, and let $T < N$ be the stabilizer of λ in N. (We have that $\mathbf{C}_{M/C}(P) = 1$ and therefore M/C has no non-trivial P-invariant irreducible characters.) Then λ extends to some $\nu \in \mathrm{IBr}(T)$ (use, for instance, [N1, Theorem 8.11]), and then $\delta = \nu^N \in \mathrm{IBr}_p(N)$ has p-power degree, not 1. Let $\varphi \in \mathrm{IBr}_p(G|\delta)$. Since φ has p-power degree, then $\varphi_N \in \mathrm{IBr}_p(N)$ by [N1, Theorem 8.30]. It follows therefore that $|N : T| = p^a$, and we conclude that the faithful action of P on $\mathrm{Irr}(M/C)$ is half-transitive. Applying the Isaacs-Passman theorem [IP], we now conclude that P is cyclic if $p > 2$. □

We remark that if $p = 2$, then already the group $G = (\mathsf{S}_3 \times \mathsf{S}_3) : C_2$ has irreducible Brauer degrees $\{1, 4\}$ and no cyclic Sylow 2-subgroup.

To finish this section, we remind the reader that the groups having exactly one non-linear irreducible Brauer character (a much stronger condition than the one in Theorem B) have recently been completely classified in [DN].

4. Proof of Theorem C

The first step in classifying non-solvable groups with two distinct degrees of irreducible 2-Brauer characters is provided by the following statement:

Proposition 4.1. *Let G be a non-solvable group with $\mathrm{cd}_2(G) = \{1, m\}$ for some $m > 1$ and let S be a non-abelian composition factor of G. Then S cannot satisfy any of the following two conditions:*

(i) *For any group H with $S \lhd H \leq \mathrm{Aut}(S)$, there exist 2-Brauer characters $\varphi, \psi \in \mathrm{IBr}_2(H)$ of distinct degrees and both not containing S in their kernels.*

(ii) *There exist non-principal 2-Brauer characters $\alpha, \beta \in \mathrm{IBr}_2(S)$ and a set π of primes such that $\alpha(1)_\pi > \beta(1)_\pi \cdot |\mathrm{Out}(S)|_\pi$.*

If in addition S is a finite simple group of Lie type in characteristic 2, then $S \cong \mathsf{A}_5$ or A_6.

Proof. By modding out a suitable normal subgroup of G, we may assume that G has a minimal normal subgroup $N = S_1 \times \ldots \times S_n$, where $S_i \cong S$ for all i.

Suppose first that S satisfies (i). Denoting $M := \mathbf{N}_G(S_1)$, $C := \mathbf{C}_G(S_1)$ and $H := M/C$, we see that $S_1 \cong NC/C \lhd H$ and $\mathbf{C}_H(NC/C) = 1$. According to (i), there are $\varphi, \psi \in \mathrm{IBr}_2(H)$ of distinct degrees and both not containing S in their kernels. By inflation, we can also view φ and ψ as irreducible Brauer characters of M. Now, let $1_{S_1} \neq \alpha_1$ be an irreducible constituent of φ_{S_1} and let

$$\alpha := \alpha_1 \times 1_{S_2} \times \ldots \times 1_{S_n} \in \mathrm{IBr}_2(N).$$

Since α_1 is non-trivial, it follows that the inertia group $I_G(\alpha)$ of α in G is contained in M. Now, φ lies over α, and by the Clifford correspondence we deduce that $\varphi^G \in \mathrm{IBr}_2(G)$, of degree $|G : M|\varphi(1) > 1$. Similarly, we obtain $\psi^G \in \mathrm{IBr}_2(G)$ of degree $|G : M|\psi(1) > 1$. But this is a contradiction since $\varphi(1) \neq \psi(1)$.

Suppose next that S satisfies (ii). We will show that S then satisfies (i). To this end, consider any H as in (i), any $\varphi \in \mathrm{IBr}_p(H)$ that lies above α, and any $\psi \in \mathrm{IBr}_2(H)$ that lies above β. Note that $H/S \leq \mathrm{Out}(S)$ is solvable. Hence by Swan's theorem [N1, Theorem 8.22], $\varphi(1) = a\alpha(1)$ with a dividing $|\mathrm{Out}(S)|$. Similarly, $\psi(1) = b\beta(1)$ with b dividing $|\mathrm{Out}(S)|$. By the choice of α and β, both φ and ψ do not contain S in their kernels, and

$$\varphi(1)_\pi \geq \alpha(1)_\pi > \beta(1)_\pi \cdot |\mathrm{Out}(S)|_\pi \geq \beta(1)_\pi \cdot b_\pi = \psi(1)_\pi.$$

Thus $\varphi(1) \neq \psi(1)$, as desired.

Finally, suppose that S is a simple group of Lie type in characteristic 2 and $S \ncong \mathsf{A}_5$, A_6. Note that ${}^2\mathrm{F}_4(2)'$ satisfies (i) (with $\varphi(1) = 26$ and $\psi(1) = 246$, cf. [JLPW]), so we also have that $S \ncong {}^2\mathrm{F}_4(2)'$. Let St denote the Steinberg character of S. It is well known that $\alpha := \mathsf{St}^0$ is irreducible, and moreover, according to the main result of [F], $\gamma := \mathsf{St} \times \ldots \times \mathsf{St} \in \mathrm{Irr}(N)$ extends to G. It follows that $\mathrm{IBr}_2(G)$ contains an irreducible character of degree $\mathsf{St}(1)^n > 1$, whence $m = \mathsf{St}(1)^n$ is a power of 2. Consider any non-principal $\delta \in \mathrm{IBr}_2(S)$ and any $\chi \in \mathrm{IBr}_2(G)$ lying above $\delta \times \ldots \times \delta \in \mathrm{IBr}_2(N)$. By the assumption on G we must have that $\chi(1) = m$ and so δ is also a 2-power. We have shown that the degree of any $\delta \in \mathrm{IBr}_2(S)$ is a 2-power.

Hence by [TW, Theorem 1.1], $S \cong \mathrm{SL}_2(q)$ with $q = 2^f \geq 8$, ${}^2\mathrm{B}_2(q)$ with $q = 2^f \geq 8$, or $\mathrm{Sp}_4(q)$ with $q = 2^f \geq 4$. In each of these cases, we have that $\alpha(1) = q$, q^2, or q^4. Accordingly, we can find $\beta \in \mathrm{IBr}_2(S)$ of degree 2, 4, or 4. Also, $|\mathrm{Out}(S)| = f$, f, or $2f$. Now setting $\pi = \{2\}$, we see that $\alpha(1)_\pi > \beta(1)_\pi \cdot |\mathrm{Out}(S)|_\pi$. Thus S satisfies (ii), a contradiction. □

Next we classify the possible non-abelian composition factors of non-solvable groups with two distinct degrees of irreducible 2-Brauer characters.

Theorem 4.2. *Let S be a finite non-abelian simple group, which does not satisfy any of the two conditions* (i) *and* (ii) *listed in Proposition 4.1. Suppose in addition that S is not isomorphic to any simple group of Lie type in characteristic* 2. *Then $S \cong \mathrm{PSL}_2(q)$ with $q = 2^a + 1 \geq 17$ a Fermat prime.*

Proof. **Case 1:** $S = \mathsf{A}_n$ with $n \geq 7$.

Consider the irreducible 2-Brauer characters α, β of S_n labeled by the partitions $(n-1, 1)$, respectively $(n-2, 2)$. As shown in the proof of [GT2, Lemma 6.1], both α and β remain irreducible over S, furthermore,

$$\alpha(1) \leq n - 1 < (n^2 - 5n)/2 \leq \beta(1)$$

(for small n this can be checked using [JLPW]). Since $\mathrm{Aut}(S) \cong \mathsf{S}_n$, it follows that S satisfies condition (i) of Proposition 4.1.

Case 2: S is a sporadic finite simple group; in particular $|\mathrm{Out}(S)| \leq 2$.

Suppose first that $S = \mathrm{M}_{12}$, M_{22}, M_{24}, J_2, HS, Ru, Suz, or Co_3. Using [JLPW] and [ModAt] we can find $\alpha, \beta \in \mathrm{IBr}_p(S)$ satisfying 4.1(ii) with $(\alpha(1), \beta(1))$ equal to

$$(10, 44),\ (34, 98),\ (44, 120),\ (36, 84),\ (20, 56),\ (28, 376),\ (142, 638),\ (22, 230),$$

respectively.

Next assume that $S = \mathrm{Co}_1$. Then it is well known that $\mathrm{IBr}_2(S)$ contains a character β of degree 24. Also, the subgroup Co_2 of Co_1 contains an irreducible complex character γ of 2-defect 0 of degree $2^{18} \cdot 7$. Choosing an irreducible constituent α in the socle of $(\gamma^\circ)^S$, we see that $\alpha(1) \geq \gamma(1)$, whence (α, β) satisfies 4.1(i).

Suppose that $S = \mathrm{BM}$. According to [J], $\mathrm{IBr}_2(S)$ contains a character β of degree 4370. Also, the subgroup $\mathrm{F}_4(2)$ of BM contains an irreducible complex character γ of 2-defect 0 of degree 2^{24}. Choosing an irreducible constituent α in the socle of $(\gamma^\circ)^S$, we see that $\alpha(1) \geq \gamma(1)$, whence (α, β) satisfies 4.1(i).

For all the other 16 sporadic simple groups, we can choose $\alpha = \chi^\circ$, where $\chi \in \mathrm{Irr}(S)$ has 2-defect 0 and $1_S \neq \beta \in \mathrm{IBr}_2(S)$ of smallest possible degree as found in [J]. It is easy to check that (α, β) satisfies 4.1(ii) (with π chosen to be the set of all primes).

Case 3. We may now assume that S is a simple group of Lie type in characteristic $r > 2$, defined over $\mathbb{F}_q$ with $q = r^f$.

Then we can find a simple, simply connected algebraic group $\mathcal{G}$ in characteristic r and a Frobenius endomorphism $F : \mathcal{G} \to \mathcal{G}$ such that $S = G/\mathbf{Z}(G)$ for $G := \mathcal{G}^F$.

Let the pair $(\mathcal{G}^*, F^*)$ be dual to $(\mathcal{G}, F)$ and let $G^* := (\mathcal{G}^*)^{F^*}$. We refer the reader to [C] and [DM] for basics on the Deligne-Lusztig theory of complex representations of finite groups of Lie type.

Recall that a **primitive prime divisor** $\mathrm{ppd}(b, n)$ for $b, n \geq 2$ is a prime divisor of $b^n - 1$ which does not divide $\prod_{i=1}^{n-1}(b^i - 1)$, cf. [Z]. In most of the cases, we will find some primitive prime divisors $\ell_i > 2$ and some semisimple ℓ_i-element $1 \neq s_i \in G^*$, such that ℓ_i is coprime to both the order of the center of the universal covering of $\mathcal{G}^*$ and $|G^* : [G^*, G^*]|$, for $i = 1, 2$. The first condition ensures that $\mathbf{C}_{\mathcal{G}^*}(s_i)$ is connected, whence one can consider the semisimple character $\chi_i = \chi_{s_i} \in \mathrm{Irr}(G)$ labeled by the G^*-conjugacy class of s_i. The second condition implies that $s_i \in [G^*, G^*]$, whence χ_i is trivial at $\mathbf{Z}(G)$ and so can be viewed as an irreducible character of S, (see e.g [NT2, Lemma 4.4]). Since $\ell_i > 2$, it follows from [HM, Proposition 1] that $\chi_i^\circ \in \mathrm{IBr}_p(S)$. Recall also that $\chi_i(1) = |G^* : \mathbf{C}_{G^*}(s_i)|_{r'}$. We will choose ℓ_1, ℓ_2 so that ℓ_1 divides $\chi_2(1)$ but not $\chi_1(1) \cdot |\mathrm{Out}(S)|$. Setting $\alpha = \chi_2^\circ$, $\beta = \chi_1^\circ$, and $\pi = \{\ell_1\}$, we will then see that S satisfies condition (ii) of Proposition 4.1.

We first illustrate this idea on the case $S = \mathrm{PSL}_n(q)$ with $n \geq 4$. Then we choose $\ell_1 = \mathrm{ppd}(r, nf) > nf$ and $\ell_2 = \mathrm{ppd}(r, (n-1)f) > (n-1)f$. Note that if $\ell_2 \leq n$, then in fact $\ell_2 = n$, in which case, since $n > 2$, we have that $\ell_2 \nmid (q-1)$ and so $\gcd(n, q-1) = 1$. In this exceptional case, $S \cong G \cong G^* \cong \mathrm{SL}_n(q)$, and by interchanging $\mathcal{G}$ with $\mathcal{G}^*$ and G with G^*, we can achieve that $\mathbf{C}_{\mathcal{G}^*}(x)$ is connected for any semisimple element $x \in G^*$. In either case, by [MT, Lemma 2.4] we can find s_i as above, with $|\mathbf{C}_{G^*}(s_1)| = (q^n - 1)/(q-1)$ and $|\mathbf{C}_{G^*}(s_2)| = q^{n-1} - 1$. Furthermore, $|\mathrm{Out}(S)|$ divides $2f \cdot \gcd(n, q-1)$ and so it is coprime to ℓ_1. Thus (s_1, s_2) has the aforementioned properties, and so S satisfies 4.1(ii).

Suppose now that $S = \mathrm{PSp}_{2n}(q)$, $\Omega_{2n+1}(q)$, or $\mathrm{P}\Omega^\epsilon_{2n}(q)$, where $n \geq 3$ and $\epsilon = \pm$. In the first two cases, using [MT, Lemma 2.4] we can choose $\ell_1 = \mathrm{ppd}(r, 2nf)$, $\ell_2 = \mathrm{ppd}(r, 2(n-1)f)$, $|\mathbf{C}_{G^*}(s_1)| = q^n + 1$, and $|\mathbf{C}_{G^*}(s_2)| = q(q^{n-1} + 1)(q^2 - 1)$. If $S = \mathrm{P}\Omega^-_{2n}(q)$, we can choose the same ℓ_1, ℓ_2 and get $|\mathbf{C}_{G^*}(s_1)| = q^n + 1$, $|\mathbf{C}_{G^*}(s_2)| = (q^{n-1}+1)(q-1)$. Assume that $S = \mathrm{P}\Omega^+_{2n}(q)$. Then, again using [MT, Lemma 2.4] we can choose $\ell_1 = \mathrm{ppd}(r, 2(n-1)f)$ and $|\mathbf{C}_{G^*}(s_1)| = (q^{n-1} + 1)(q + 1)$. Furthermore,

$$(\ell_2, |\mathbf{C}_{G^*}(s_2)|) = \begin{cases} (\mathrm{ppd}(r, nf), q^n - 1), & n \text{ odd}, \\ (\mathrm{ppd}(r, (n-1)f), (q^{n-1} - 1)(q - 1)), & n \text{ even}. \end{cases}$$

Assume that $S = \mathrm{PSU}_n(q)$ with $n \geq 5$ (note that the case $\mathrm{PSU}_4(q) \cong \mathrm{P}\Omega^-_6(q)$ has already been considered). Using [MT, Lemma 2.3] we can choose

$$(\ell_1, \ell_2) = \begin{cases} (\mathrm{ppd}(r, 2nf), \mathrm{ppd}(r, 2(n-2)f)), & n \text{ odd}, \\ (\mathrm{ppd}(r, 2(n-1)f), \mathrm{ppd}(r, 2(n-3)f)), & n \text{ even}. \end{cases}$$

Suppose that $S = \mathrm{G}_2(q)$, ${}^2\mathrm{G}_2(q)$, ${}^3\mathrm{D}_4(q)$, $\mathrm{F}_4(q)$, $\mathrm{E}_6(q)$, ${}^2\mathrm{E}_6(q)$, $\mathrm{E}_7(q)$, or $\mathrm{E}_8(q)$. Then using [MT, Lemma 2.3] we can choose $\ell_i = \mathrm{ppd}(r, m_i f)$ for $i = 1, 2$, with

$$(m_1, m_2) = (6,3),\ (6,1),\ (12,6),\ (12,8),\ (12,9),\ (12,18),\ (18,14),\ (30,24),$$

respectively. In all the above cases, we have seen that S satisfies 4.1(ii).

Assume that $S = \mathrm{PSp}_4(q)$ with $q \geq 3$. Then according to [W, Theorem 3.1], there are $\alpha, \beta \in \mathrm{IBr}_2(S)$ with $\alpha(1) = q(q-1)^2/2$ and $\beta(1) = (q^2-1)/2$. Hence S satisfies 4.1(ii) with $\pi = \{r\}$.

Suppose now that $S = \mathrm{PSU}_3(q)$. It is well known (see e.g. [Ge]) that there is $\alpha \in \mathrm{IBr}_2(S)$ with $\alpha(1) = q(q-1)$. Choosing $\ell_2 = \mathrm{ppd}(r, 6f)$ and arguing as above, we obtain $\beta = \chi_2^\circ \in \mathrm{IBr}_2(S)$ of degree $(q^2-1)(q+1)$. Thus S satisfies 4.1(ii) with $\pi = \{r\}$.

Next suppose that $S = \mathrm{PSL}_3(q)$. It is well known (see e.g. [GT1]) that there is $\alpha \in \mathrm{IBr}_2(S)$ with $\alpha(1) = q(q+1)$. Choosing $\ell_2 = \mathrm{ppd}(r, 3f)$ and arguing as above, we obtain $\beta = \chi_2^\circ \in \mathrm{IBr}_2(S)$ of degree $(q^2-1)(q-1)$. Thus S satisfies 4.1(ii) with $\pi = \{r\}$.

Finally, we consider the case $S = \mathrm{PSL}_2(q)$. Suppose that $q-1$ is not a 2-power. Then, according to [B], there are $\alpha, \beta \in \mathrm{IBr}_2(S)$ with $\alpha(1) = (q-1)/2$ and $\beta(1) = q+1$. It is straightforward to check that S satisfies 4.1(ii) for a suitably chosen π. Hence we conclude that $q = 2^a + 1$ and so $q \geq 17$ is a Fermat prime (note that $\mathrm{PSL}_2(5) \cong \mathrm{SL}_2(4)$ and $\mathrm{PSL}_2(9) \cong \mathrm{Sp}_4(2)'$). $\square$

Proof of Theorem C. Suppose G is any non-solvable group with $\mathbf{O}_2(G) = 1$ and $\mathrm{cd}_2(G) = \{1, m\}$, where $m > 1$. Then G admits a non-abelian composition factor S. By Proposition 4.1 and Theorem 4.2, $S \cong \mathrm{PSL}_2(q)$ with $q = 2^a + 1 \geq 5$ either a Fermat prime or 9. Letting $L := G^{(\infty)}$, without loss we may assume that there is a chief factor $L/K \cong S^n$ of G (for some n). It is well known that $\mathrm{IBr}_2(S)$ consists of 1_S, exactly two characters α_1, α_2 of degree $(q-1)/2 = 2^{a-1}$, and 2^{a-2} characters $\beta_1, \ldots, \beta_{2^{a-2}}$ of degree $q-1$ (see e.g. [B]). Furthermore, $\mathrm{Out}(S)$ has order 4 if $q = 9$ and 2 otherwise.

Step 1. First we show that $n = 1$, i.e. $L/K \cong S$, and $m = q - 1 = 2^a$.

Indeed, working in G/K instead of G, we may assume that G has a minimal normal subgroup $N = S_1 \times \ldots \times S_n$ with $S_i \cong S$. Consider the character

$$\alpha = \alpha_1 \times 1_{S_2} \times \ldots \times 1_{S_n} \in \mathrm{IBr}_2(N).$$

Then the inertia subgroup $J := I_G(\alpha)$ contains $C_1 := \mathbf{C}_G(S_1)$ (as a normal subgroup) and is contained in $M := \mathbf{N}_G(S_1)$. In particular, $J/C_1S_1 \leq M/C_1S_1 \leq \mathrm{Out}(S)$ is a 2-group. Working in M/C_1 and using Green's theorem [N1, Theorem 8.11], we see that α extends to a character $\hat{\alpha}$ of J. Also, since S has exactly two irreducible Brauer characters of degree $(q-1)/2$, $c := |M : J| \leq 2$. Hence by the Clifford correspondence, $\hat{\alpha}^G \in \mathrm{IBr}_2(G)$ and has degree $2^{a-1}cn$. By the hypothesis, $m = 2^{a-1}cn$.

On the other hand, $\mathrm{IBr}_2(G)$ also contains a character lying above $\beta_1 \times \beta_1 \times \ldots \times \beta_1 \in \mathrm{IBr}_2(N)$, of degree divisible by $\beta_1(1)^n = 2^{an}$. In particular,

$$2^a n \geq 2^{a-1}cn = m \geq 2^{an}.$$

Since $a \geq 2$, it follows that $n = 1$, $c = 2$, and $m = 2^a$.

Step 2. Let $C \lhd G$ be such that $C/K = \mathbf{C}_G(L/K)$, and let $\tau \in \mathrm{IBr}_2(C)$ be any character of degree > 1. Then τ extends to CL.

Suppose that the inertia subgroup $I_{CL}(\tau)$ of τ in CL has index d in CL. Then, by the Clifford correspondence, $\mathrm{IBr}_2(CL)$ contains a character of degree divisible by d. In turn, the same is true for $\mathrm{IBr}_2(G)$ since $CL \lhd G$. Hence d divides $m = 2^a$. But $I_{CL}(\tau) \rhd C$ and $CL/C \cong S \cong \mathrm{PSL}_2(q)$ has no proper subgroup of index dividing 2^a. It follows that $d = 1$, i.e. τ is CL-invariant.

By [N1, Theorem 8.28], the modular character triple (CL, C, τ) is **isomorphic** (as defined in [N1, Definition 8.25]) to a modular character triple (Y, Z, λ), where $Y/Z \cong CL/C \cong S$, $Z \leq \mathbf{Z}(Y)$ is of odd order, and λ is a faithful linear character of Z. If $q \neq 9$, then since the Schur multiplier of S has order 2, we must have that $Y = Y^{(\infty)} \times Z$ with $Y^{(\infty)} \cong S$. In this case, λ extends to Y and so τ extends to CL. The same is true when $q = 9$, unless $Y^{(\infty)} \cong 3\mathsf{A}_6$. In the latter case, any irreducible Brauer character of Y lying above λ has degree divisible by 3. It follows that any irreducible Brauer character μ of CL lying above τ also has degree divisible by 3. But then any irreducible Brauer character ν of G lying above μ also has degree divisible by 3, contradicting the fact that such ν must have degree $m = 2^a$.

Step 3. C, and so K, is abelian of odd order.

Suppose that $\mathrm{IBr}_2(C)$ contains a character τ of degree > 1. By Step 3, τ extends to a character φ of CL. Hence, if we view β_1 as an irreducible Brauer character of $CL/C \cong S$, then by [N1, Corollary 8.20], $\varphi\beta_1 \in \mathrm{IBr}_2(CL)$. It follows that $\mathrm{IBr}_2(G|\varphi\beta_1)$ contains a character of degree $\geq (\varphi\beta_1)(1) = (q-1)\tau(1) > m$, a contradiction. We have shown that $\mathrm{cd}_2(C) = \{1\}$. Furthermore, $\mathbf{O}_2(C) = 1$ as $\mathbf{O}_2(G) = 1$. It follows that C is abelian of odd order.

Step 4. $K = 1$ and $CL = C \times L \cong C \times S$.

Consider any $\lambda \in \mathrm{IBr}_2(K) = \mathrm{Irr}(K)$. Arguing as in Step 2, we see that λ extends to L. But $L = G^{(\infty)}$ is perfect, hence $\lambda = 1_C$. Thus $K = 1$, $L \cong S$, $C \cap L \leq \mathbf{Z}(L) = 1$, whence $CL = C \times L$.

Step 5. $C = \mathbf{Z}(G)$, $G/CL \cong C_2$, and G induces the group of inner-diagonal automorphisms of S.

Consider any $\mu \in \mathrm{IBr}_2(C) = \mathrm{Irr}(C)$ and the character $\beta_j \times \mu \in \mathrm{IBr}_2(CL)$ of degree $q - 1 = m$. Since any member of $\mathrm{IBr}_2(G|\beta_j \times \mu)$ must have degree m, we see that $\beta_j \times \mu$ is G-invariant. In particular, G fixes every irreducible character of the abelian group C, and so $C \leq \mathbf{Z}(G)$. On the other hand, $\mathbf{Z}(G) \leq \mathbf{C}_G(L) = C$, whence $C = \mathbf{Z}(G)$. We also have that $G/CL \hookrightarrow \mathrm{Out}(S)$ and so G/CL is a 2-group.

Next we consider the character $\alpha_1 \times 1_C \in \mathrm{IBr}_2(CL)$ of degree $(q-1)/2$. By Green's theorem, it extends to its inertia group T in G, and so we get an irreducible Brauer character of degree $|G : T| \cdot m/2$. Certainly, this degree must be equal to m, and so $|G : T| = 2$. Now if $q \neq 9$, then $|\mathrm{Out}(S)| = 2$ and we conclude that $G/CL \cong \mathrm{Out}(S) \cong C_2$.

Suppose that $q = 9$, and so $\mathrm{Out}(S) \cong C_2^2$. We already know that G/CL has order at least 2, and G fixes each β_j. Note that each of the two involutions 2_1 and 2_3 (in the notation of [Atlas]) of $\mathrm{Out}(S)$ interchanges β_1 and β_2. Hence $G/CL = \langle 2_2 \rangle$. Thus in all cases G induces the group of inner-diagonal automorphisms of S.

We have proved the 'only if' direction of Theorem C.

Step 6. The 'if' direction of Theorem C holds.

Suppose G has the structure described in Theorem C. Then G interchanges α_1 and α_2 (the two irreducible Brauer characters of degree $(q-1)/2$ of S), and fixes all the other irreducible Brauer characters of S (of degree 1 and $q-1$). Since $G/\mathbf{Z}(G)S$ has order 2 and $\mathbf{Z}(G)$ is central of odd order, it is easy to check that $\mathrm{cd}_2(G) = \{1, q-1\}$.

We have completed the proof of Theorem C.

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