Nested Quantifiers

Section 1.5

Section Summary

- Nested Quantifiers
- Order of Quantifiers
- Translating to and from English
- Negating Nested Quantifiers

Nested Quantifiers

- Two quantifiers are nested if one is within the scope of the other.

Example: "Every real number has an inverse" is

$$\forall x \underbrace{\exists y \overbrace{(x + y = 0)}^{P(x,y)}}_{Q(x)}$$

$\forall x\, Q(x)$
$Q(x)$ is $\exists y\, P(x,y)$
$P(x,y)$ is "$x+y=0$"

where the domains of x and y are the real numbers.

Thinking of Nested Quantification as Nested Loops

Loop through all values of x. At each step, loop through all values of y.

- $\forall x \forall y P(x,y)$
  - If $P(x,y)$ is false for some pair of x and y, then $\forall x \forall y P(x,y)$ is false and both the outer and inner loop terminate.
  - $\forall x \forall y\, P(x,y)$ is true if the outer loop ends after stepping through each x.
- $\forall x \exists y P(x,y)$
  - The inner loop ends when a pair x and y is found such that $P(x, y)$ is true.
  - If no y is found such that $P(x, y)$ is true, the outer loop terminates as $\forall x \exists y P(x,y)$ has been shown to be false.
  - $\forall x \exists y P(x,y)$ is true if the outer loop ends after stepping through each x.

Order of Quantifiers

Examples:

1. $P(x,y)$: "$x + y = y + x$." Assume that U is the real numbers.
   - $\forall x \forall y P(x,y)$ and $\forall y \forall x P(x,y)$ have the same truth value.
   - $\exists x \exists y P(x,y)$ and $\exists y \exists x P(x,y)$ have the same truth value.

2. $Q(x,y)$: "$x + y = 0$." Assume that U is the real numbers.
   - $\forall x \exists y Q(x,y)$ is true, but
   - $\exists y \forall x Q(x,y)$ is false

Questions on Order of Quantifiers

Example 1: Let U be the real numbers,

Define $P(x,y)$: "$x \cdot y = 0$"

What is the truth value of the following:

1. $\forall x \forall y P(x,y)$

   Answer: False
2. $\forall x \exists y P(x,y)$

   Answer: True
3. $\exists x \forall y\, P(x,y)$

   Answer: True
4. $\exists x\, \exists\, y\, P(x,y)$

   Answer: True

Questions on Order of Quantifiers

Example 2: Let *U* be the positive numbers,
Define *P(x,y)* : "*x* / *y* = 1"
What is the truth value of the following:

1. $\forall x \forall y P(x,y)$
   Answer: False
2. $\forall x \exists y P(x,y)$
   Answer: True
3. $\exists x \forall y\, P(x,y)$
   Answer: False
4. $\exists x\, \exists y\, P(x,y)$
   Answer: True

Quantifications of Two Variables

Statement	When True?	When False
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	*P*(*x*,*y*) is true for every pair *x*,*y*.	There is a pair *x*, *y* for which *P*(*x*,*y*) is false.
$\forall x \exists y P(x,y)$	For every *x* there is a *y* for which *P*(*x*,*y*) is true.	There is an x such that *P*(*x*,*y*) is false for every *y*.
$\exists x \forall y P(x,y)$	There is an *x* for which *P*(*x*,*y*) is true for every *y*.	For every *x* there is a y for which *P*(x,y) is false.
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair *x*, *y* for which *P*(*x*,*y*) is true.	*P*(x,y) is false for every pair *x*,*y*

Translating Nested Quantifiers into English

Let U be all students in your school. Using C(x)="x has a computer," and $F(x,y)$ ="x and y are friends," translate the following statements.

- $\forall x \ (C(x) \lor \exists y \ (C(y) \land F(x, y)))$

Solution: Every student in your school has a computer or has a friend who has a computer.

- $\exists x \ \forall y \ \forall z \ ((F(x, y) \land F(x,z) \land (y \neq z)) \rightarrow \neg F(y,z))$

Solution: There is a student none of whose friends are also friends with each other.

Translating Mathematical Statements into Predicate Logic

1. Rewrite the statement to make the implied quantifiers and domains explicit
2. Introduce variables and specify the domain for them
3. Rewrite the statement using quantifiers, variables, and logic expressions.

Example : Translate "The sum of two positive integers is always positive" into a logical expression.

1. "For every two positive integers, the sum of these integers is positive."
2. "For all positive integers x and y, $x + y$ is positive."
3. $\forall x \forall y (x + y > 0)$, where the domain of both variables consists of all positive integers

Translating English into Logical Expressions Example

Example: Use quantifiers to express the statement "There is a woman who has taken a flight on every airline."

Solution:

- Let $P(w,f)$ = "w has taken f" and $Q(f,a)$ = "f is a flight on a."
- The domain of w is all women, the domain of f is all flights, and the domain of a is all airlines.
- Then the statement can be expressed as:

$$\exists w\, \forall a\, \exists f\, (P(w,f) \land Q(f,a))$$

Questions on Translation from English

Choose the obvious predicates and express in predicate logic.

Example 1: "Brothers are siblings."

Solution: $\forall x\ \forall y\ (B(x,y) \rightarrow S(x,y))$

Example 2: "Siblinghood is symmetric."

Solution: $\forall x\ \forall y\ (S(x,y) \rightarrow S(y,x))$

Example 3: "Everybody loves somebody."

Solution: $\forall x\ \exists y\ L(x,y)$

Example 4: "There is someone who is loved by everyone."

Solution: $\exists y\ \forall x\ L(x,y)$

Example 5: "There is someone who loves someone."

Solution: $\exists x\ \exists y\ L(x,y)$

Example 6: "Everyone loves himself"

Solution: $\forall x\ L(x,x)$

Negating Nested Quantifiers

Example 1: Express the negation of the statement $\forall x \exists y(xy=1)$ so that no negation precedes a quantifier.

Solution: Use De Morgan's Laws to move the negation as far inwards as possible.

1. $\neg\forall x \exists y(xy = 1)$
2. $\exists x \neg\exists y(xy = 1)$ by De Morgan's for $\forall$
3. $\exists x \forall y \neg(xy = 1)$ by De Morgan's for $\exists$
4. $\exists x \forall y(xy \neq 1)$

Negating Nested Quantifiers

Translate the following statement into a logical expression.

"There does not exist a woman who has taken a flight on every airline."

Solution:

- Translate the positive sentence into a logical expression
  - $\exists w \forall a \exists f\ (P(w,f) \land Q(f,a))$ [by previous example]
  - $P(w,f)$: "w has taken f" $Q(f,a)$: "f is a flight on a."
- Find the negation of the logical expression
  - $\neg\exists w \forall a \exists f\ (P(w,f) \land Q(f,a))$
  - $\forall w \neg \forall a \exists f\ (P(w,f) \land Q(f,a))$
  - $\forall w \exists a \neg \exists f\ (P(w,f) \land Q(f,a))$
  - $\forall w \exists a \forall f \neg (P(w,f) \land Q(f,a))$
  - $\forall w \exists a \forall f (\neg P(w,f) \lor \neg Q(f,a))$

"For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline"