Rules of Inference

Section 1.6

Section Summary

- Valid Arguments
- Inference Rules for Propositional Logic
  - Building Arguments
- Inference Rules for Quantified Statements
  - Building Arguments

Revisiting the Socrates Example

- We have the two premises:
  - "All men are mortal."
  - "Socrates is a man."
- And the conclusion:
  - "Socrates is mortal."
- How do we get the conclusion from the premises?

The Argument

- We can express the premises (above the line) and the conclusion (below the line) in predicate logic as an argument:

$$\forall x(Man(x) \rightarrow Mortal(x)) \quad \text{premise}$$

$$Man(Socrates) \quad \text{premise}$$

$$\therefore \quad Mortal(Socrates) \quad \text{conclusion}$$

- We will see shortly that this is a valid argument.

Arguments in Propositional Logic

- A *argument* in propositional logic is a sequence of propositions. All but the final proposition are called *premises*. The last statement is the *conclusion*.
- The argument is *valid* if the premises imply the conclusion.
  - If the premises are $p_1, p_2, \ldots, p_n$ and the conclusion is q then $(p_1 \land p_2 \land \ldots \land p_n) \rightarrow q$ is a tautology.
- An *argument form* in a propositional logic is an argument that is valid no matter what propositions are substituted into its propositional variables.
- Inference rules are all simple argument forms that will be used to construct more complex argument forms.

Rules of Inference for Propositional Logic:

Modus Ponens

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

Corresponding Tautology:

$(p \wedge (p \rightarrow q)) \rightarrow q$

Example:

Let p be "It is snowing."
Let q be "I will study discrete math."

"If it is snowing, then I will study discrete math."
"It is snowing."

"Therefore , I will study discrete math."

Modus Tollens

$$\begin{array}{l} p \rightarrow q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

Corresponding Tautology:

$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$

Example:

Let p be "It is snowing."
Let q be "I will study discrete math."

"If it is snowing, then I will study discrete math."
"I will not study discrete math."

"Therefore , it is not snowing."

Hypothetical Syllogism

Corresponding Tautology:

$((p \to q) \wedge (q \to r)) \to (p \to r)$

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

Example:

Let p be "it snows."
Let q be "I will study discrete math."
Let r be "I will get an A."

"If it snows, then I will study discrete math."
"If I study discrete math, I will get an A."

"Therefore , If it snows, I will get an A."

Disjunctive Syllogism

Corresponding Tautology:

$(\neg p \wedge (p \vee q)) \rightarrow q$

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

Example:

Let p be “I will study discrete math.”

Let q be “I will study English literature.”

“I will study discrete math or I will study English literature.”

“I will not study discrete math.”

“Therefore , I will study English literature.”

Addition

$$\frac{p}{\therefore p \vee q}$$

Corresponding Tautology:

$p \rightarrow (p \vee q)$

Example:

Let p be “I will study discrete math.”
Let q be “I will visit Las Vegas.”

“I will study discrete math.”

“Therefore, I will study discrete math or I will visit Las Vegas.”

Simplification

$$\frac{p \wedge q}{\therefore q}$$

Corresponding Tautology:

$(p \wedge q) \rightarrow q$

Example:

Let p be “I will study discrete math.”
Let q be “I will study English literature.”

“I will study discrete math and English literature”

“Therefore, I will study discrete math.”

Conjunction

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

Corresponding Tautology:
$((p) \wedge (q)) \rightarrow (p \wedge q)$

Example:
Let p be "I will study discrete math."
Let q be "I will study English literature."

"I will study discrete math."
"I will study English literature."

"Therefore, I will study discrete math and I will study English literature."

Resolution

Resolution plays an important role in AI and is used in Prolog.

$$\begin{array}{r} p \vee q \\ \neg p \vee r \\ \hline \therefore\ q \vee r \end{array}$$

Corresponding Tautology:
$((\neg p \vee r) \wedge (p \vee q)) \rightarrow (q \vee r)$

Example:
Let *p* be "I will study discrete math."
Let *r* be "I will study English literature."
Let q be "I will study databases."

"I will not study discrete math or I will study English literature."
"I will study discrete math or I will study databases."

"Therefore, I will study databases or I will English literature."

Arguments that use rules of inference

- Example1 : State which rule of inference is the basis of the following argument:
  - “It is below freezing now.
  - Therefore, it is either below freezing or raining now.”
- *Solution*: P:”It is below freezing now” and q “It is raining now.”
- Then this argument is of the form:

$$\begin{array}{l} \quad p \\ \hline \therefore p \vee q \end{array}$$

- This is an argument that uses the addition rule.

Arguments that use rules of inference

- Example2 : State which rule of inference is used in the argument:
  - If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have a barbecue tomorrow.
- Solution: P:" It is raining today," q: "We will not have a barbecue today," and r:"We will have a barbecue tomorrow."
- Then this argument is of the form:

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore\ p \to r \end{array}$$

- This is an argument that uses the hypothetical syllogism.

Using the Rules of Inference to Build Valid Arguments

- A *valid argument* is a sequence of statements. Each statement is either a premise or follows from previous statements by rules of inference. The last statement is called conclusion.
- A valid argument takes the following form:

$$
\begin{array}{l}
S_1 \\
S_2 \\
\cdot \\
\cdot \\
\cdot \\
S_n \\
\\
\therefore C
\end{array}
$$

Valid Arguments: Example 1

Example: From the single proposition

$$p \land (p \rightarrow q)$$

Show that q is a conclusion.

Solution:

Step	**Reason**
1. $p \land (p \rightarrow q)$	Premise
2. p	Simplification using (1)
3. $p \rightarrow q$	Simplification using (1)
4. q	Modus Ponens using (2) and (3)

Valid Arguments: Example 2

Example: With these hypotheses: $\neg p \wedge q$

- "It is not sunny this afternoon and it is colder than yesterday."
- "We will go swimming only if it is sunny." $r \rightarrow p$
- "If we do not go swimming, then we will take a canoe trip." $\neg r \rightarrow s$
- "If we take a canoe trip, then we will be home by sunset." $s \rightarrow t$

Using the inference rules, construct a valid argument for the conclusion: "**We will be home by sunset.**" t

Solution:

p : "It is sunny this afternoon." r : "We will go swimming."
q : "It is colder than yesterday." s : "We will take a canoe trip."
t : "We will be home by sunset."

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Valid Arguments: Example 2

Hypotheses: $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, $s \rightarrow t$
Conclusion: t

Step	**Reason**
1. $\neg p \wedge q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \rightarrow p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \rightarrow s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \rightarrow t$	Premise
8. t	Modus ponens using (6) and (7)

Handling Quantified Statements

- Valid arguments for quantified statements are a sequence of statements. Each statement is either a premise or follows from previous statements by rules of inference which include:
  - Rules of Inference for Propositional Logic
  - Rules of Inference for Quantified Statements
- The rules of inference for quantified statements are introduced in the next several slides.

Rules of Inference for Quantified Statements:

TABLE 2 Rules of Inference for Quantified Statements.

Rule of Inference	*Name*
$\forall x P(x)$ $\therefore P(c)$	Universal instantiation
$P(c)$ for an arbitrary c $\therefore \forall x P(x)$	Universal generalization
$\exists x P(x)$ $\therefore P(c)$ for some element c	Existential instantiation
$P(c)$ for some element c $\therefore \exists x P(x)$	Existential generalization

Universal Instantiation (UI)

$$\frac{\forall x P(x)}{\therefore P(c)}$$

Example:

Our domain consists of all women and Lisa is a woman.

All women are wise”

“Therefore, Lisa is wise”

Universal Generalization (UG)

$$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$$

Used often implicitly in Mathematical Proofs.

Example:

"If $x > y$, where x and y are positive real numbers, then $x^2 > y^2$"
"For all positive real numbers x and y, if $x > y$, then $x^2 > y^2$"

Existential Instantiation (EI)

$$\frac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$$

Example:

"There is someone who got an A in the course."
"Let's call her *a* and say that *a* got an A"

Existential Generalization (EG)

$$\frac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$$

Example:

"Michelle got an A in the class."
"Therefore, someone got an A in the class."

Using Rules of Inference: Example 1

Ex: Using the rules of inference, construct a valid argument to show that "John Smith has two legs" is a consequence of the premises:

"Every man has two legs." and "John Smith is a man."

Solution: Let $M(x)$ denote "x is a man" and $L(x)$ "x has two legs" and let John Smith be a member of the domain.

Step	**Reason**
1. $\forall x(M(x) \to L(x))$	Premise
2. $M(J) \to L(J)$	UI from (1)
3. $M(J)$	Premise
4. $L(J)$	Modus Ponens using (2) and (3)

Using Rules of Inference: Example 2

Ex: Use the rules of inference to construct a valid argument showing that the conclusion "Someone who passed the first exam has not read the book." follows from the premises

"A student in this class has not read the book."

"Everyone in this class passed the first exam."

Solution: Let $C(x)$ denote "x is in this class," $B(x)$ denote "x has read the book," and $P(x)$ denote "x passed the first exam."

$$\begin{array}{l} \exists x(C(x) \land \neg B(x)) \\ \forall x(C(x) \rightarrow P(x)) \\ \hline \therefore\ \exists x(P(x) \land \neg B(x)) \end{array}$$

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Using Rules of Inference: Example 2

Valid Argument:

Step	**Reason**
1. $\exists x(C(x) \land \neg B(x))$	Premise
2. $C(a) \land \neg B(a)$	EI from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \to P(x))$	Premise
5. $C(a) \to P(a)$	UI from (4)
6. $P(a)$	MP from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \land \neg B(a)$	Conj from (6) and (7)
9. $\exists x(P(x) \land \neg B(x))$	EG from (8)

Returning to the Socrates Example

$$\begin{array}{l} \forall x(Man(x) \rightarrow Mortal(x)) \\ Man(Socrates) \\ \hline \therefore \quad Mortal(Socrates) \end{array}$$

Valid Argument

Step	**Reason**
1. $\forall x(Man(x) \rightarrow Mortal(x))$	Premise
2. $Man(Socrates) \rightarrow Mortal(Socrates)$	UI from (4)
3. $Man(Socrates)$	Premise
4. $Mortal(Socrates)$	MP from (2) and (3)

Universal Modus Ponens

Universal Modus Ponens combines universal instantiation and modus ponens into one rule.

$$\begin{array}{l} \forall x(P(x) \rightarrow Q(x)) \\ P(a), \text{where } a \text{ is a particular} \\ \quad \text{element in the domain} \\ \hline \therefore Q(a) \end{array}$$

This rule could be used in the Socrates example.

Additional Examples

Determine whether the argument is correct or incorrect.

- "Everyone majoring in computer science has Linux installed." $\forall x(C(x) \rightarrow L(x))$
- "George doesn't have Linux installed." $\neg L(George)$
- "Therefore, George isn't majoring in computer science." $\neg C(George)$

$$\begin{array}{l} \forall x(C(x) \rightarrow L(x)) \\ \neg L(George) \\ \hline \therefore \neg C(George) \end{array}$$

Correct!
Universal Modus Tollens

Determine whether the argument is correct or incorrect.

- A Dvorak keyboard is efficient to use. $\forall x(D(x) \rightarrow E(x))$
- Jake's keyboard is not a Dvorak keyboard. $\neg D(Jake)$
- Therefore, Jake's keyboard is not efficient. $\neg E(Jake)$

Incorrect! We can't conclude ¬E(j) with this information

$$\begin{array}{l} \forall x(D(x) \rightarrow E(x)) \\ \neg D(j) \\ \hline \therefore \neg E(j) \end{array}$$

Prove the following hypothesis implies the conclusion "It rained" r

- "If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on." $(\neg r \lor \neg f) \rightarrow (s \land l)$
- "If the sailing race is held, then the trophy will awarded." $s \rightarrow t$
- "The trophy was not awarded." $\neg t$

f="It's foggy." s="The sailing race is held." r="It rains." t="The trophy is awarded."

l="The life saving demonstrations will go on."

$$(\neg r \vee \neg f) \rightarrow (s \wedge l)$$
$$s \rightarrow t$$
$$\neg t$$
$$\therefore r$$

Step	Reason
1. $\neg t$	Premise
2. $s \rightarrow t$	Premise
3. $\neg s$	Modus Tollens using (1) and (2)
4. $(\neg r \vee \neg f) \rightarrow (s \wedge l)$	Premise
5. $\neg s \vee \neg l$	Addition using (3)
6. $\neg(s \wedge l)$	De Morgan's law using (5)
7. $\neg(\neg r \vee \neg f)$	Modus Tollens using (4) and (6)
8. $r \wedge f$	De Morgan's law using (7)
9. r	Simplification using (8)