Set Operations

Section 2.2

Section Summary

- Set Operations
  - Union
  - Intersection
  - Complementation
  - Difference
- Inclusion-Exclusion Principle
- Set Identities
- Proving Identities
- Membership Tables

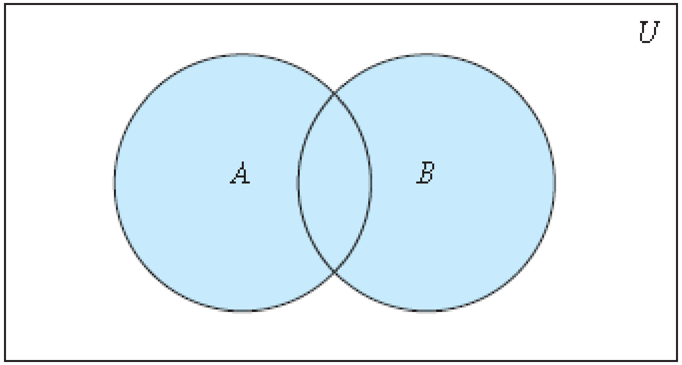
Boolean Algebra

- Propositional logic (Sometimes called Propositional calculus) and set theory are both instances of an algebraic system called *Boolean Algebra*.
- The operators in set theory are analogous to the corresponding operator in propositional logic.
- As always there must be a universal set U. All sets are assumed to be subsets of U.

Union

- **Definition**: Let A and B be sets. The *union* of the sets A and B, denoted by $A \cup B$, is the set:

$$\{x | x \in A \vee x \in B\}$$



$A \cup B$ is shaded.

- **Ex**: What is $\{1,2,3\} \cup \{3, 4, 5\}$?

  Solution: $\{1,2,3,4,5\}$

Intersection

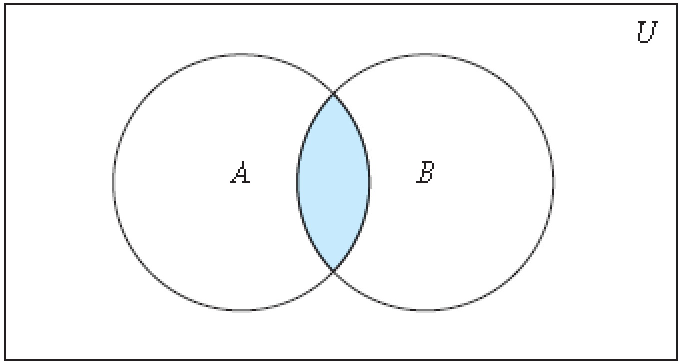
- **Definition**: The *intersection* of sets *A* and *B*, denoted by $A \cap B$, is

$$\{x | x \in A \land x \in B\}$$

- Note if the intersection is empty, then *A* and *B* are said to be *disjoint*.
- **Ex**: What is {1,2,3} ∩ {3,4,5}?

  Solution: {3}
- **Ex:** What is {1,2,3} ∩ {4,5,6}?

  Solution: ∅

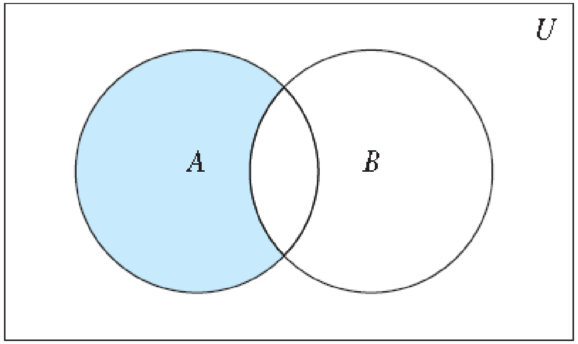


$A \cap B$ is shaded.

Difference

- **Definition**: Let A and B be sets. The *difference* of A and B, denoted by $A - B$ ***(Sometimes A\B),*** is the set containing the elements of A that are not in B.
- The difference of A and B is also called the complement of B with respect to A.

$$A - B = \{x \mid x \in A \land x \notin B\} = A \cap \overline{B}$$



$A - B$ is shaded.

Complement

Definition: If A is a set, then the *complement* of A (with respect to U), denoted by $\bar{A}$ is the set $U - A$

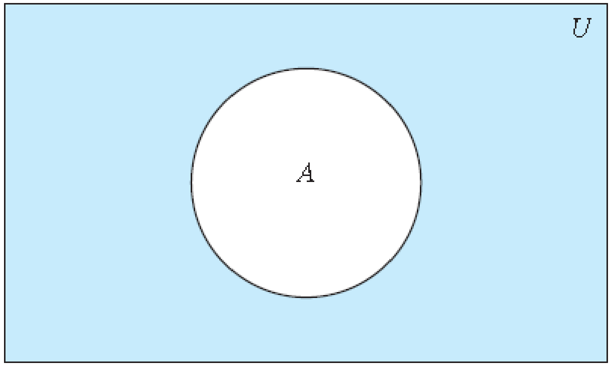
$$\bar{A} = \{x \in U \mid x \notin A\}$$

(The complement of A is sometimes denoted by A^c.)

Ex: If U is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$

Solution: $\{x \mid x \leq 70\}$

Venn Diagram for Complement

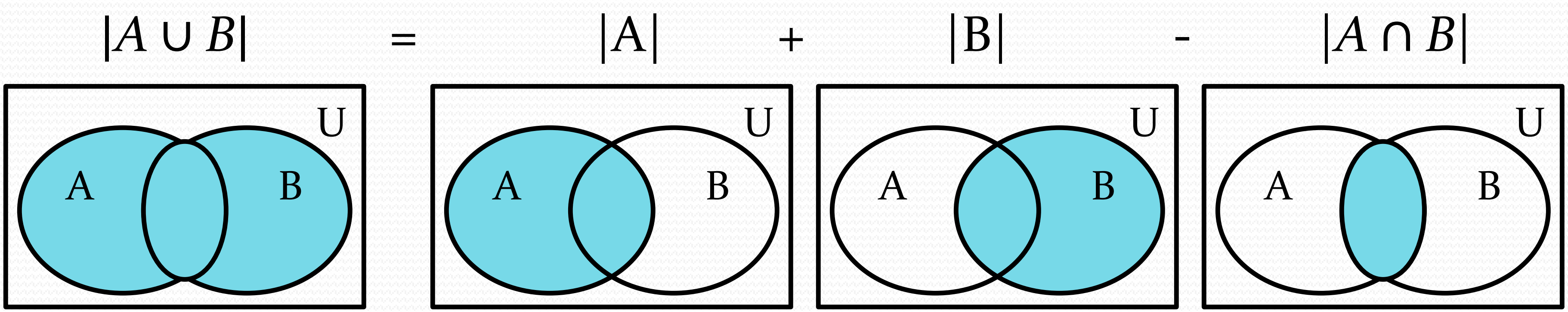


$\bar{A}$ is shaded.

The Cardinality of the Union of Two Sets

- Inclusion-Exclusion

$|A \cup B| = |A| + |B| - |A \cap B|$



- **Ex**: Let A be the math majors in your class and B be the CS majors. To count the number of students who are either math majors or CS majors, add the number of math majors and the number of CS majors, and subtract the number of joint CS/math majors.

Review Questions

Example: $U = \{0,1,2,3,4,5,6,7,8,9,10\}$ $A = \{1,2,3,4,5\}$, $B = \{4,5,6,7,8\}$

1. $A \cup B$

   Solution: $\{1,2,3,4,5,6,7,8\}$
2. $A \cap B$

   Solution: $\{4,5\}$
3. $\bar{A}$

   Solution: $\{0,6,7,8,9,10\}$
4. $\bar{B}$

   Solution: $\{0,1,2,3,9,10\}$
5. $A - B$

   Solution: $\{1,2,3\}$
6. $B - A$

   Solution: $\{6,7,8\}$

Symmetric Difference (*optional*)

Definition: The *symmetric difference* of **A** and **B**, denoted by $A \oplus B$ is the set

$$(A - B) \cup (B - A)$$

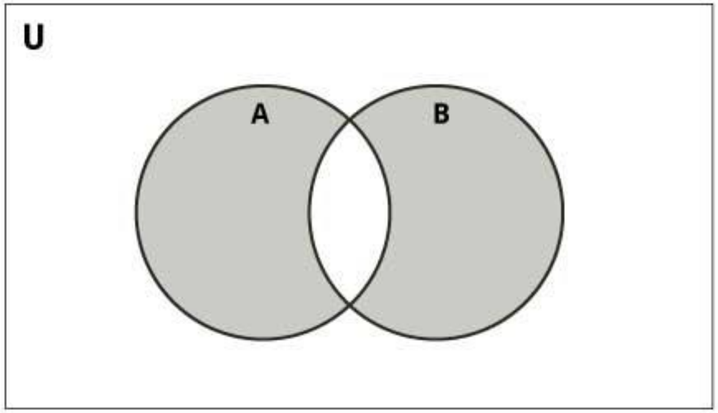
Example:

$U = \{0,1,2,3,4,5,6,7,8,9,10\}$

$A = \{1,2,3,4,5\}$ $B = \{4,5,6,7,8\}$

What is $A \oplus B$

- **Solution**: {1,2,3,6,7,8}



Set Identities

- Identity laws

$$A \cup \emptyset = A \qquad A \cap U = A$$

- Domination laws

$$A \cup U = U \qquad A \cap \emptyset = \emptyset$$

- Idempotent laws

$$A \cup A = A \qquad A \cap A = A$$

- Complementation law

$$\overline{(\overline{A})} = A$$

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Set Identities

- Commutative laws

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

- Associative laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

- Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

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Set Identities

- De Morgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

- Absorption laws

$$A \cup (A \cap B) = A \qquad A \cap (A \cup B) = A$$

- Complement laws

$$A \cup \overline{A} = U \qquad A \cap \overline{A} = \emptyset$$

Proving Set Identities A=B

Different ways to prove set identities:

1. Produce a series of identical sets beginning with A and ending with B.
2. Prove that each set (side of the identity) is a subset of the other.
3. Use set builder notation and propositional logic.
4. Membership Tables: Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity. Use 1 to indicate it is in the set and a 0 to indicate that it is not.

Proving Set Identity

- Ex: Prove that $\overline{\left((A \cap B) \cup \bar{B}\right)} = B \cap \bar{A}$
- Solution:

$$
\begin{aligned}
\overline{\left((A \cap B) \cup \bar{B}\right)} &= \overline{(A \cap B)} \cap \bar{\bar{B}} && \textit{by De Morgan} \\
&= \overline{(A \cap B)} \cap B && \textit{by Double Complement} \\
&= (\bar{A} \cup \bar{B}) \cap B && \textit{by De Morgan} \\
&= (\bar{A} \cap B) \cup (\bar{B} \cap B) && \textit{by Distributivity} \\
&= (\bar{A} \cap B) \cup \emptyset && \textit{by Complement} \\
&= (\bar{A} \cap B) && \textit{by Identity} \\
&= B \cap \bar{A} && \textit{by Commutative} \quad \blacktriangleleft
\end{aligned}
$$

Proof of Second De Morgan Law

Ex: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Solution: We prove this identity by showing that each side is a subset of the other:

1) $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and

2) $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

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Proof of Second De Morgan Law

These steps show that: $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

$x \in \overline{A \cap B}$	by assumption
$x \notin A \cap B$	defn. of complement
$\neg((x \in A) \wedge (x \in B))$	defn. of intersection
$\neg(x \in A) \vee \neg(x \in B)$	1st De Morgan Law for Prop Logic
$x \notin A \vee x \notin B$	defn. of negation
$x \in \overline{A} \vee x \in \overline{B}$	defn. of complement
$x \in \overline{A} \cup \overline{B}$	defn. of union

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Proof of Second De Morgan Law

These steps show that: $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

$x \in \overline{A} \cup \overline{B}$	by assumption
$(x \in \overline{A}) \vee (x \in \overline{B})$	defn. of union
$(x \notin A) \vee (x \notin B)$	defn. of complement
$\neg(x \in A) \vee \neg(x \in B)$	defn. of negation
$\neg((x \in A) \wedge (x \in B))$	by 1st De Morgan Law for Prop Logic
$\neg(x \in A \cap B)$	defn. of intersection
$x \in \overline{A \cap B}$	defn. of complement

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Set-Builder Notation: Second De Morgan Law

Ex: Prove (again) that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Solution: We show this using set builder notion and propositional logic

$$
\begin{aligned}
\overline{A \cap B} &= \{x | x \notin A \cap B\} && \text{by defn. of complement} \\
&= \{x | \neg(x \in (A \cap B))\} && \text{by defn. of does not belong symbol} \\
&= \{x | \neg(x \in A \wedge x \in B\} && \text{by defn. of intersection} \\
&= \{x | \neg(x \in A) \vee \neg(x \in B)\} && \text{by 1st De Morgan law} \\
& && \text{for Prop Logic} \\
&= \{x | x \notin A \vee x \notin B\} && \text{by defn. of not belong symbol} \\
&= \{x | x \in \overline{A} \vee x \in \overline{B}\} && \text{by defn. of complement} \\
&= \{x | x \in \overline{A} \cup \overline{B}\} && \text{by defn. of union} \\
&= \overline{A} \cup \overline{B} && \text{by meaning of notation}
\end{aligned}
$$

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Membership Table

Example: Construct a membership table to show that the distributive law holds.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Solution:

A	B	C	B∩C	A∪(B∩C)	A∪B	A∪C	(A∪B)∩(A∪C)
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

Generalized Unions and Intersections

- Let A_1, A_2 ,...., A_n be an indexed collection of sets.

  We define:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n \qquad \bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

  These are well defined, since union and intersection are associative.

- For i = 1,2,...., let A_i = {i, i + 1, i + 2, ...}. Then,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{1, 2, 3, \ldots\}$$

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{n, n+1, n+2, \ldots..\} = A_n$$