# Functions

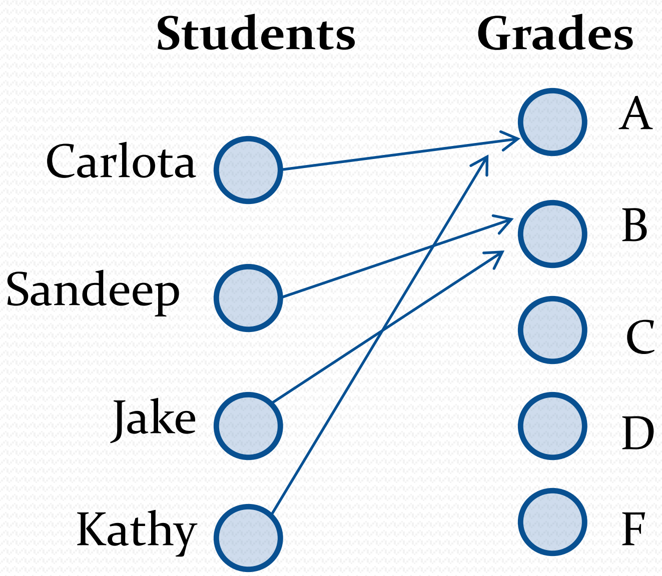
Section 2.3

# Section Summary

- Definition of a Function.
  - Domain, Codomain
  - Image, Preimage
- Injection, Surjection, Bijection
- Inverse Function
- Function Composition
- Graphing Functions
- Floor, Ceiling, Factorial, Modulo

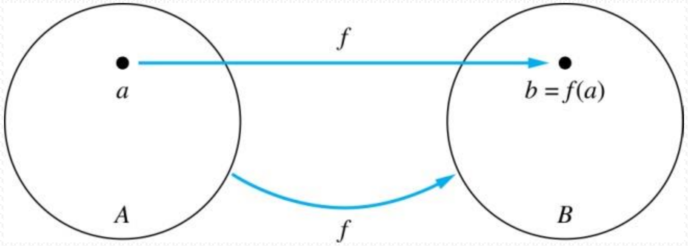
# Functions

- **Definition**: Let $A$ and $B$ be nonempty sets. A *function* $f$ from $A$ to $B$ (*denoted as* $f : A \rightarrow B$) is an assignment of exactly one element of $B$ to each element of $A$.
- We write $f(a) = b$ if $b$ is the unique element of $B$ assigned by the function $f$ to the element $a$ of $A$.




- Functions are sometimes called *mappings* or *transformations*.
- A function $f : A \rightarrow B$ can be one to one or many to one but not one to many.

# Given a function $f: A \to B$




- We say $f$ *maps* $A$ to $B$ or
  $f$ is a *mapping* from $A$ to $B$.
- $A$ is called the *domain* of $f$.
- $B$ is called the *codomain* of $f$.
- If $f(a) = b$,
  - then $b$ is called the *image* of $a$ under $f$.
  - $a$ is called the *preimage* of $b$.
- The *range* of $f$ is the set of all images of points in **A** under $f$. We denote it by $f(A)$.
- Two functions are *equal* when they have the same domain, the same codomain, and map each element of the domain to the same element of the codomain.

# Representing Functions

Functions may be specified in different ways:

- An explicit statement of the assignment.

  Students and grades example.

- A formula.

  $f(x) = x + 1$

- A computer program.

  A C++ program that when given an integer $n$, produces the $n$th Fibonacci Number (covered in the next section and also in Ch. 5).

# Questions

$f(a) = ?$ z

The image of $d$ is ? z

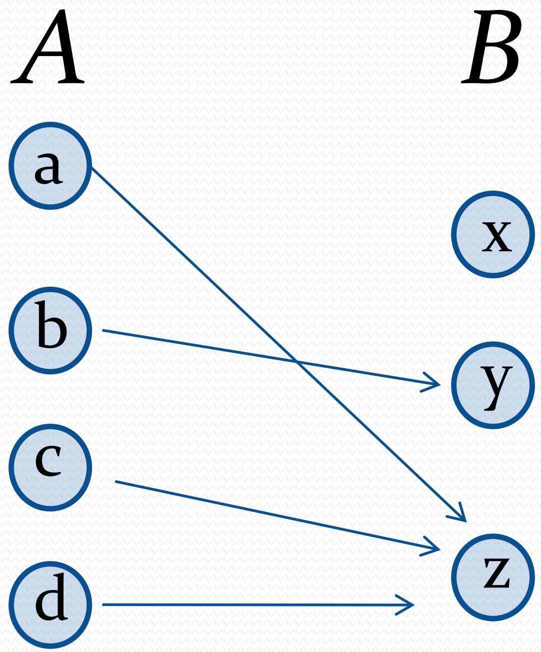
The domain of $f$ is ? $A$

The codomain of $f$ is ? $B$

The preimage of $y$ is ? b

$f(A) = ?$ {y,z}

The preimage(s) of $z$ is (are) ? {a,c,d}



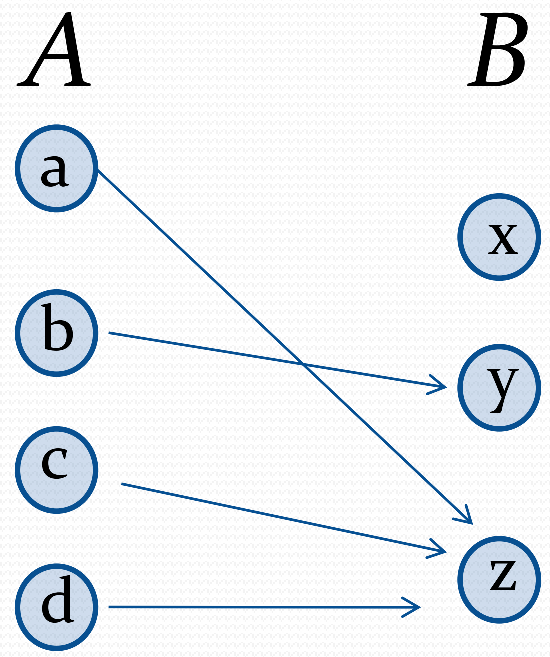
# Question on Functions and Sets

- If $f : A \rightarrow B$ and S is a subset of A, then

$$f(S) = \{f(s) | s \in S\}$$

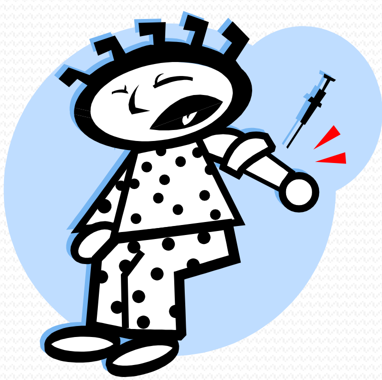
$f(\{a,b,c\})$ is ? {y,z}

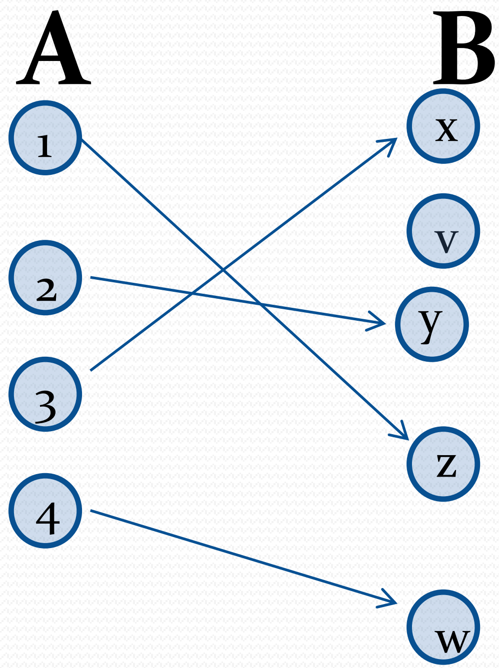
$f(\{c,d\})$ is ? {z}



# Injections

**Definition**: A function f is said to be *one-to-one*, or *injective*, if and only if $f(a) = f(b)$ implies that $a = b$ for all $a$ and $b$ in the domain of $f$. A function is said to be an *injection* if it is one-to-one.



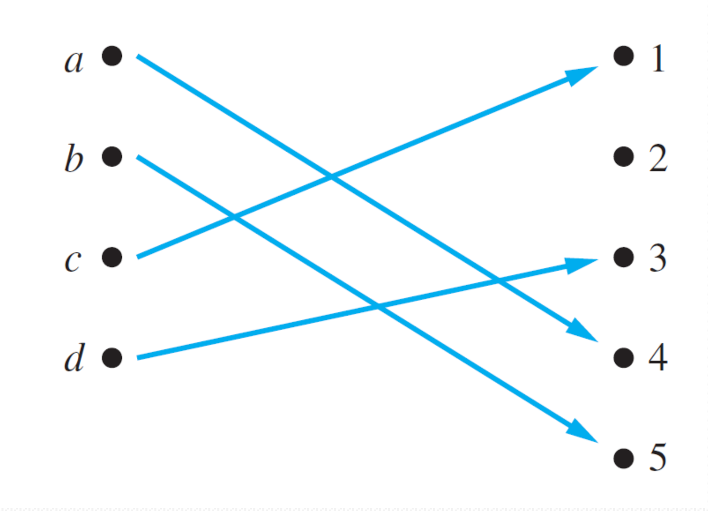


# Injections

- Using quantifiers to express on-to-one function
  - $\forall a \forall b (f(a) = f(b) \rightarrow a = b)$
  - or equivalently $\forall a \forall b (a \neq b \rightarrow f(a) \neq f(b))$,
  - where the domain of the quantifiers is the domain of the function.

# Injections

- Ex1: Determine whether the function $f$ from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, and $f(d) = 3$ is one-to-one.
- *Solution:* The function $f$ is one-to-one because $f$ takes on different values at the four elements of its domain.



# Injections

**Ex2** : Is the function $f: Z \rightarrow Z, where\ f(x) = x^2$ on-to-one?

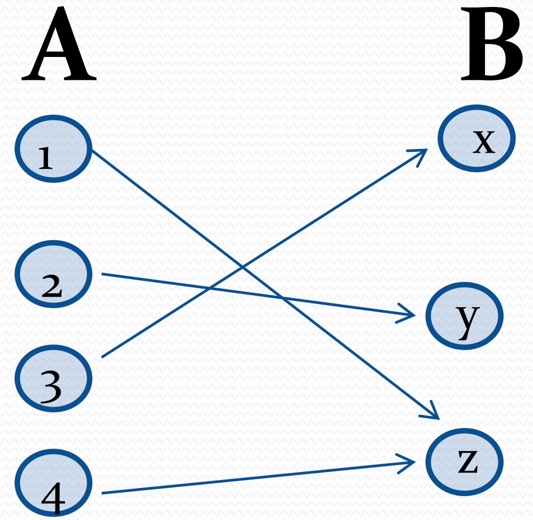
- **Solution**: No, $f$ is not on-to-one because, for instance, $f(1) = f(-1) = 1$, but $1 \neq -1$.

**Ex3** : Is the function $f: Z^+ \rightarrow Z$, where $f(x) = x^2$ on-to-one?

- **Solution**: yes, $f$ is on-to-one

# Surjections

**Definition**: A function $f$ from $A$ to $B$ is called *onto* or *surjective*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function $f$ is called a *surjection* if it is onto.

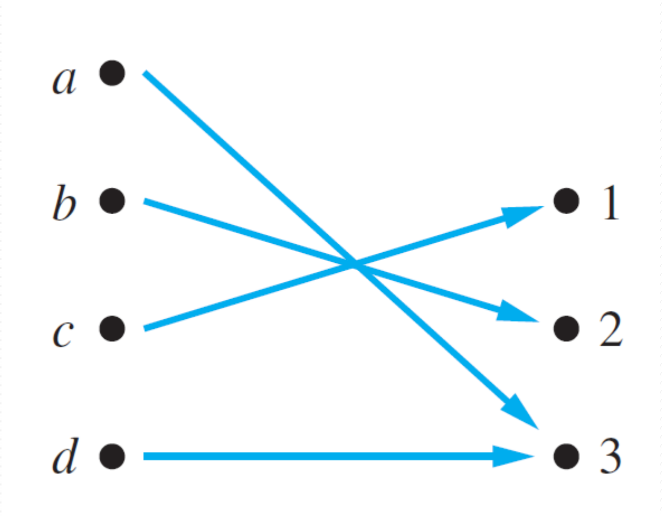


# surjections

- Using quantifiers to express onto function
  - $\forall b \exists a (f(a) = b)$
  - where the domain for a is the domain of the function and the domain for b is the codomain of the function.

# Surjections

- **Ex1**: Determine whether the function $f$ from $\{a, b, c, d\}$ to $\{1, 2, 3,\}$ with $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$ is onto.
- ***Solution***: Because all three elements of the codomain are images of elements in the domain, we see that $f$ is onto.



# Surjections

**Ex2** : Is the function $f\colon Z \rightarrow Z, where\ f(x) = x^2$ onto?

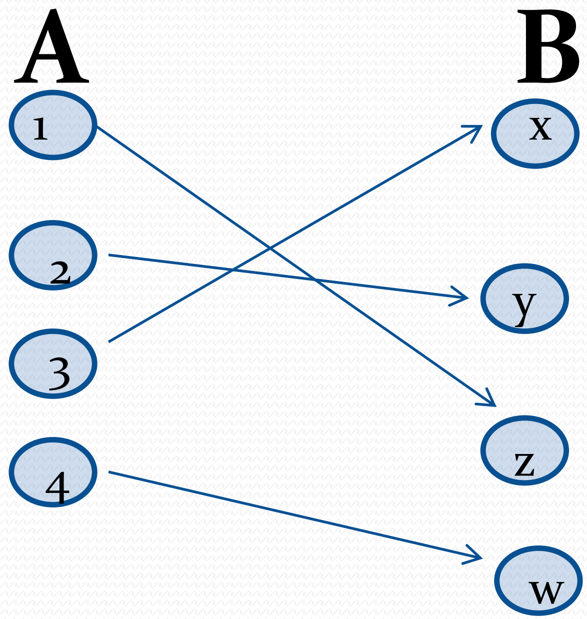
- **Solution**: The function f is not onto because there is no integer x with $x^2 = -1$, for instance.

**Ex3** : Is the function $f\colon Z^+ \rightarrow Z^+$, where $f(x) = x^2$ onto?

- **Solution**: The function f is not onto because there is no integer x with $x^2 = 2$, for instance.

# Bijections

**Definition**: A function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto (injective and surjective).



# Bijections

**Ex** : Is the function $f$: $\mathbf{Z} \rightarrow \mathbf{Z}$, where $f(x) = x^2$ bijective?

**Solution**: No, because it is neither injective nor surjective.

# Showing that *f* is injective or surjective

Suppose that $f : A \to B$.

*To show that $f$ is injective* Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ with $x \neq y$, then $x = y$.

*To show that $f$ is not injective* Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

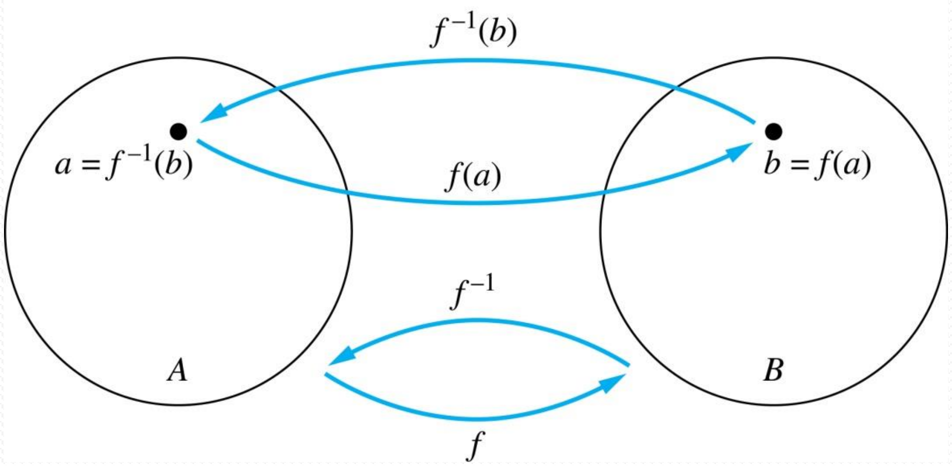
*To show that $f$ is surjective* Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

*To show that $f$ is not surjective* Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

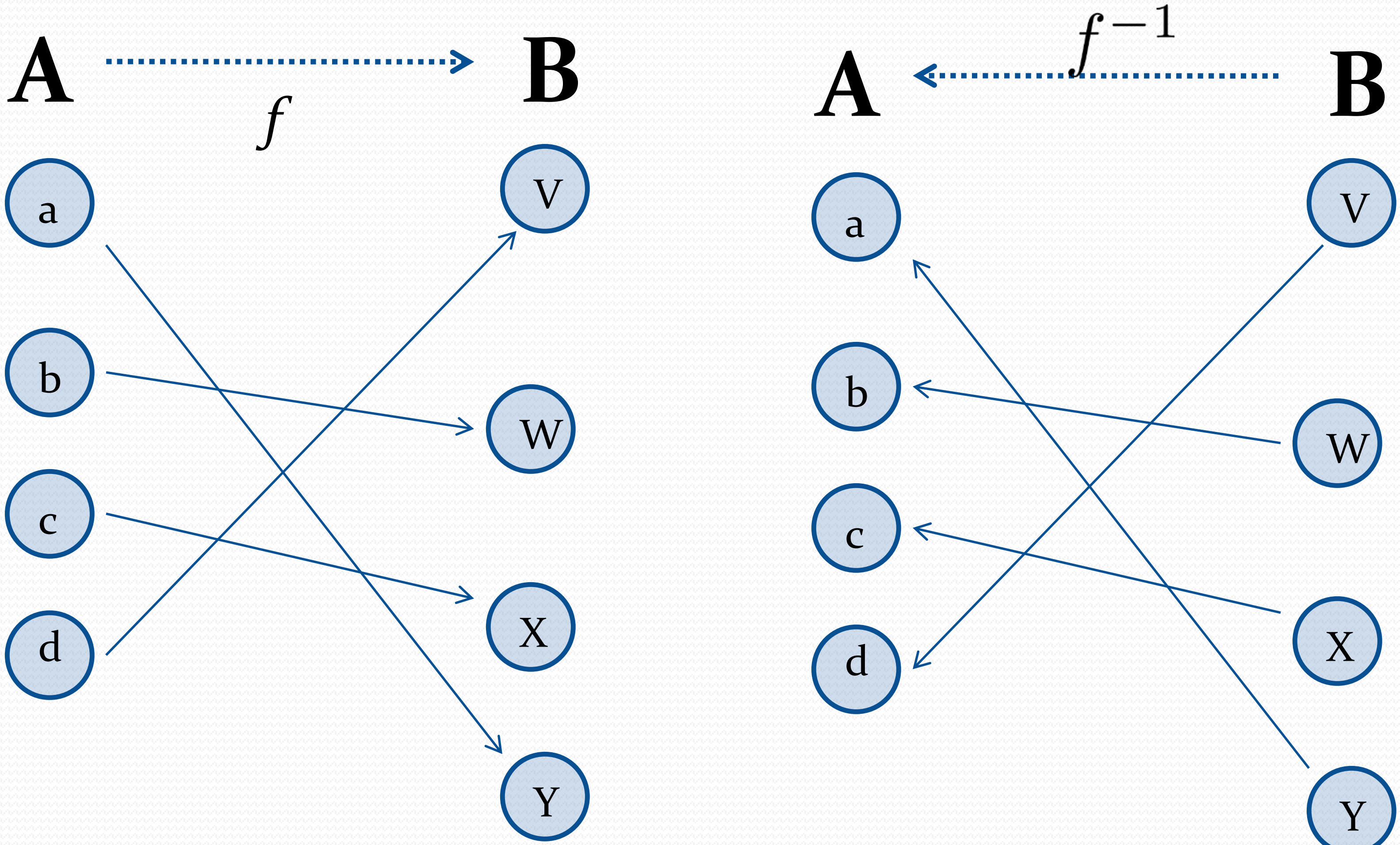
# Inverse Functions

**Definition**: Let $f$ be a bijection from $A$ to $B$. Then the *inverse* of $f$, denoted $f^{-1}$, is the function from $B$ to $A$ defined as $f^{-1}(y) = x \text{ iff } f(x) = y$

No inverse exists unless $f$ is a bijection. Why?



# Inverse Functions



# Questions

**Ex 1**: Let $f$ be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is $f$ invertible and if so what is its inverse?

**Solution**: The function $f$ is invertible because it both injective and surjective. The inverse function $f^{-1}$ reverses the correspondence given by $f$, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

# Questions

**Ex 2**: Let $f$: $\mathbf{Z} \rightarrow \mathbf{Z}$ be such that $f(x) = x + 1$. Is $f$ invertible, and if so, what is its inverse?

**Solution**: The function $f$ is invertible because it is a bijection. The inverse function $f^{-1}$ reverses the correspondence so $f^{-1}(y) = y - 1$.

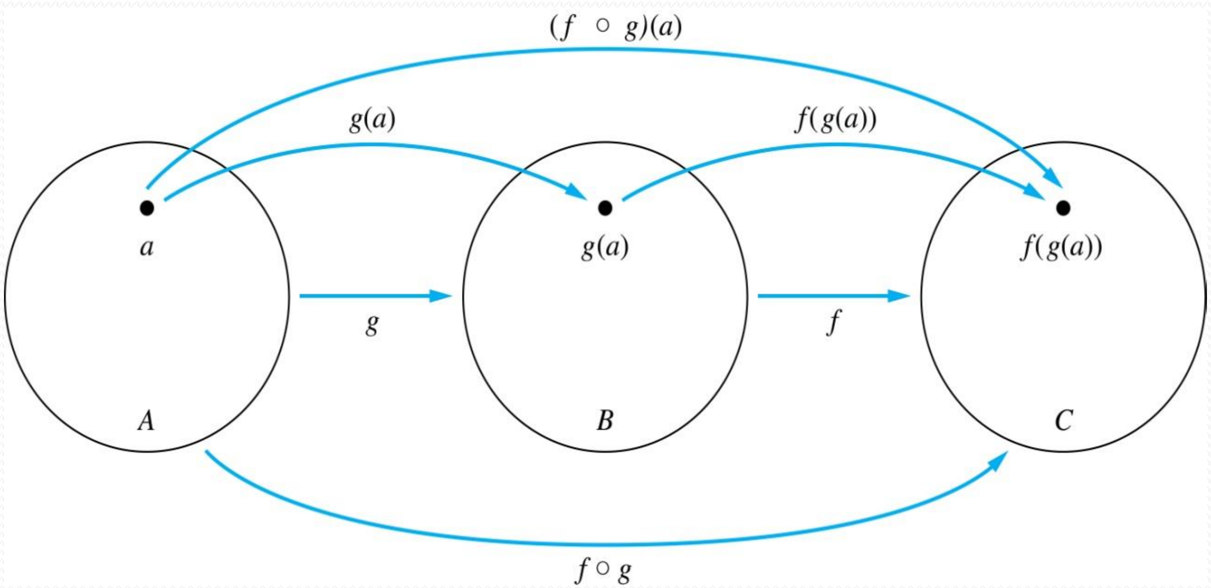
# Questions

**Ex 3**: Let $f$: **R** $\rightarrow$ **R** be such that $f(x) = x^2$. Is $f$ invertible, and if so, what is its inverse?

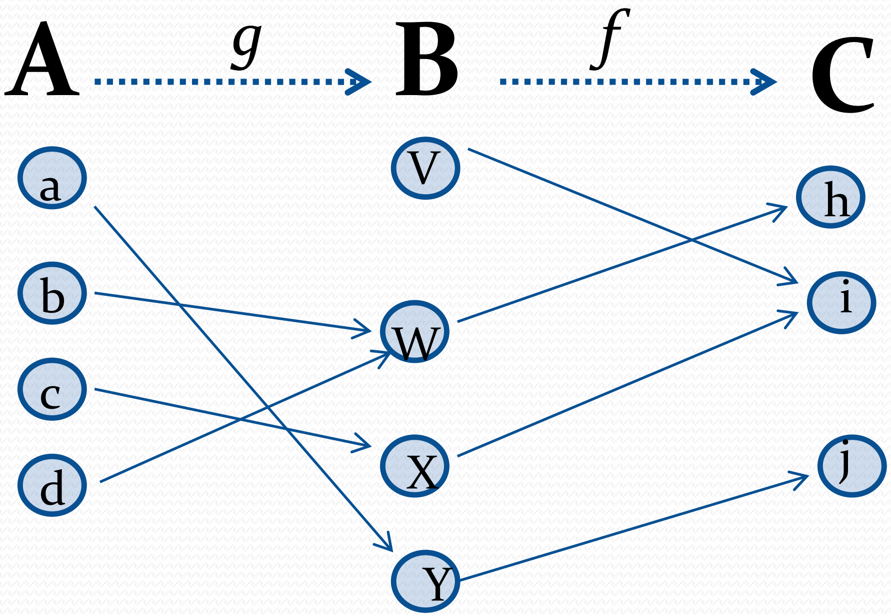
**Solution**: The function $f$ is not invertible because it is not surjective (nor injective).

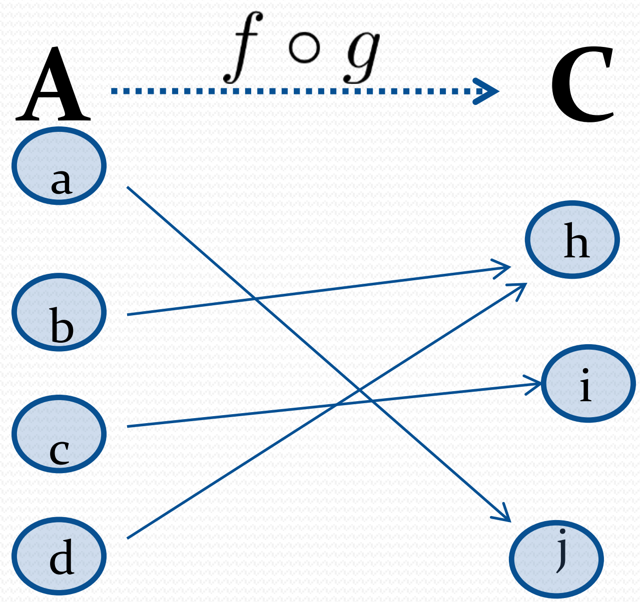
# Composition

- **Definition**: Let $f: B \to C$, $g: A \to B$. The *composition of f with g*, denoted $f \circ g$ is the function from $A$ to $C$ defined by $$(f \circ g)(a) = f(g(a))$$



# Composition



# Composition Questions

**Ex 1**: If $f(x) = x^2$ and $g(x) = 2x + 1$, then

and $$f(g(x)) = (2x+1)^2$$

$$g(f(x)) = 2x^2 + 1$$

# Composition Questions

**Ex 2**: Let $g$ be the function from the set $\{a,b,c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let $f$ be the function from the set $\{a,b,c\}$ to the set $\{1,2,3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.

What is the composition of $f$ with $g$, and what is the composition of $g$ with $f$.

**Solution:** The composition $f \circ g$ is defined by

$$f \circ g\ (a) = f(g(a)) = f(b) = 2.$$

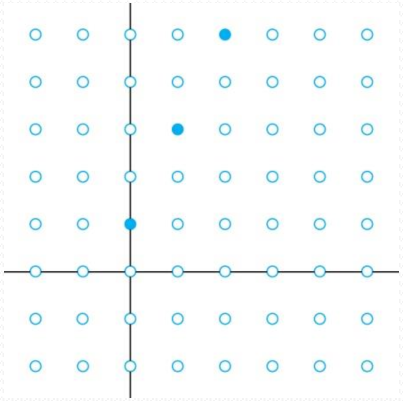
$$f \circ g\ (b) = f(g(b)) = f(c) = 1.$$

$$f \circ g\ (c) = f(g(c)) = f(a) = 3.$$

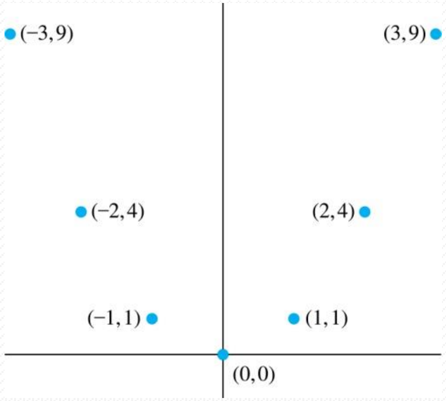
Note that $g \circ f$ is not defined, because the range of $f$ is not a subset of the domain of $g$.

# Graphs of Functions

- Let $f$ be a function from the set $A$ to the set $B$. The *graph* of the function $f$ is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a) = b\}$.



Graph of $f(n) = 2n + 1$ from Z to Z




Graph of $f(x) = x^2$ from Z to Z

# Some Important Functions

- The *floor* function $f: R \to Z$, denoted $f(x) = \lfloor x \rfloor$ is the largest integer less than or equal to *x*.

- The *ceiling* function $f: R \to Z$, denoted $f(x) = \lceil x \rceil$ is the smallest integer greater than or equal to *x*

**Ex:**

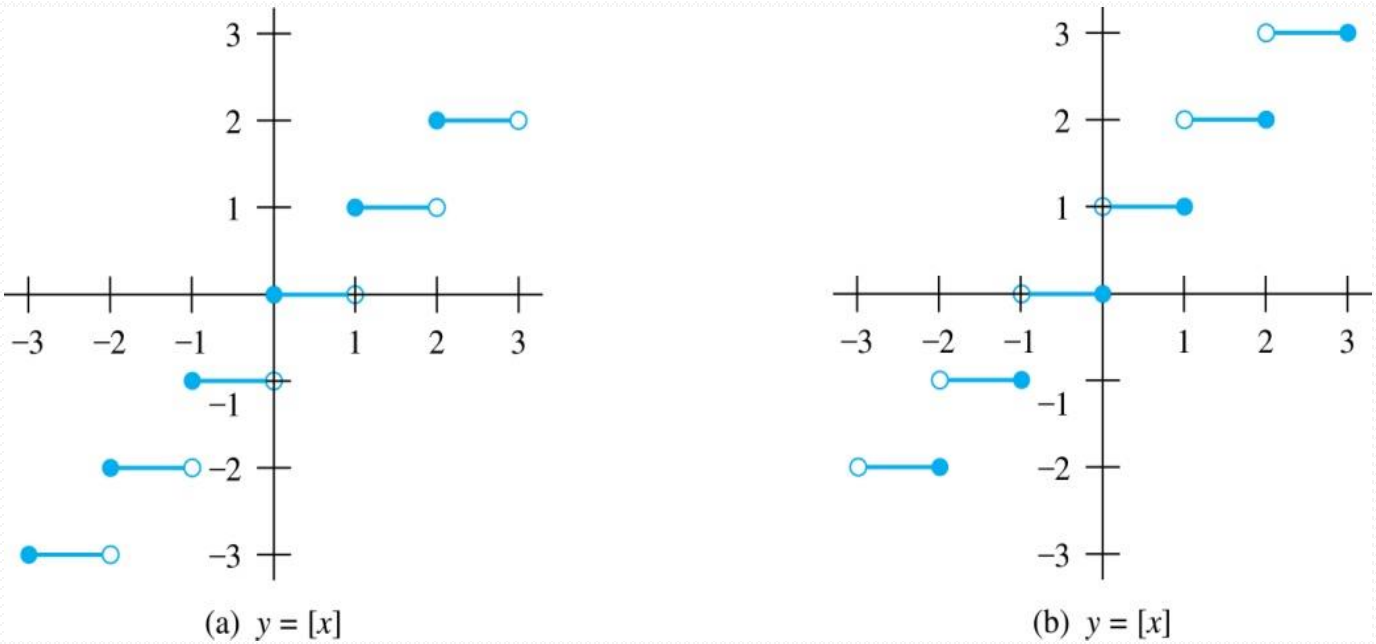
$\lfloor 3.5 \rfloor = 3$ $\qquad$ $\lceil 3.5 \rceil = 4$

$\lfloor 2.1 \rfloor = 2$ $\qquad$ $\lceil 2.1 \rceil = 3$

$\lfloor -1.5 \rfloor = -2$ $\qquad$ $\lceil -1.5 \rceil = -1$

$\lfloor -1.1 \rfloor = -2$ $\qquad$ $\lceil -1.1 \rceil = -1$

# Floor and Ceiling Functions




Graph of (a) Floor and (b) Ceiling Functions

# Floor and Ceiling Functions

**TABLE 1 Useful Properties of the Floor and Ceiling Functions.**
($n$ is an integer, $x$ is a real number)

| | |
|---|---|
| (1a) | $\lfloor x \rfloor = n$ if and only if $n \le x < n + 1$ |
| (1b) | $\lceil x \rceil = n$ if and only if $n - 1 < x \le n$ |
| (1c) | $\lfloor x \rfloor = n$ if and only if $x - 1 < n \le x$ |
| (1d) | $\lceil x \rceil = n$ if and only if $x \le n < x + 1$ |
| (2) | $x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$ |
| (3a) | $\lfloor -x \rfloor = -\lceil x \rceil$ |
| (3b) | $\lceil -x \rceil = -\lfloor x \rfloor$ |
| (4a) | $\lfloor x + n \rfloor = \lfloor x \rfloor + n$ |
| (4b) | $\lceil x + n \rceil = \lceil x \rceil + n$ |

# Factorial Function

**Definition:** The factorial function $f$: $\mathbf{N} \rightarrow \mathbf{Z}^{+}$ , denoted by $f(n) = n!$ is the product of the first $n$ positive integers when $n$ is a nonnegative integer.

$$f(n) = 1 \cdot 2 \cdots (n - 1) \cdot n, \qquad f(0) = 0! = 1$$

**Examples:**

$$f(1) = 1! = 1$$

$$f(2) = 2! = 1 \cdot 2 = 2$$

$$f(6) = 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$$

$$f(20) = 2{,}432{,}902{,}008{,}176{,}640{,}000.$$

Stirling's Formula:

$$n! \sim \sqrt{2\pi n}(n/e)^n$$

$$f(n) \sim g(n) \doteq lim_{n\rightarrow\infty} f(n)/g(n) = 1$$

# Modulo Function

**Definition:** If $a$ is an integer and $d$ is a positive integer, then there are unique integers $q$ and $r$ with $0 \leq r < d$, such that

$a = dq + r.$

The modulo function, denoted ***a mod d*** **is the remainder** ***r*** when $a$ is divided by $d$.

**Examples:**

| | |
|---|---|
| 16 mod 12 = 4 | (16 = 12 * 1 + 4) |
| 14 mod 4 = 2 | (14 = 4 * 3 + 2) |
| 5 mod 2 = 1 | ( 5 = 2 * 2 + 1) |
| 1 mod 3 = 1 | (1 = 3* 0 + 1) |
| 3 mod 7 = 4 | (3 = 7* 0 + 3) |
| 9 mod 3 = 0 | ( 9 = 3 * 3 + 0) |
| -5 mod 3 = 1 | ( -5 = 3 *- 2 + 1) |
| -17 mod 5 = 3 | (-17= 5 * -4 + 3) |