Sequences and Summations Section 2.4

Section Summary

- Sequences
 - Ex: Geometric Progression, Arithmetic Progression
- Recurrence Relations
 - Ex: Fibonacci Sequence
- Summations

Introduction

- Sequences are ordered lists of elements.
 - 1, 2, 3, 5, 8
 - 1, 3, 9, 27, 81,
- Sequences arise throughout mathematics, computer science, and in many other disciplines, ranging from botany to music.
- We will introduce the terminology to represent sequences and sums of the terms in the sequences.

Sequences

- **Definition**: A *sequence* is a function from a subset of the integers (usually either the set {0, 1, 2, 3, 4,} or {1, 2, 3, 4,}) to a set *S*.
- We use the notation $\{a_n\}$ to describe the sequence.
 - *a_n* represents an individual term of the sequence {*an*}
- The notation a_n is used to denote the image of the integer n. We can think of a_n as the equivalent of f(n) where f is a function from {0,1,2,....} to S.

Sequences

Example: Consider the sequence $\{a_n\}$ where $a_n = \frac{1}{n}$

$$\{a_n\} = a_1, a_2, a_3, a_4, \dots$$

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \dots$$

Geometric Progression

Definition: A *geometric progression* is a sequence of the form: $a, ar, ar^2, \ldots, ar^n, \ldots$

where the *initial term a* and the *common ratio r* are real numbers. **Ex**:

1. Let
$$a = 1$$
 and $r = -1$. Then:

$$\{b_n\} = b_0, b_1, b_2, b_3, b_4, \dots = 1, -1, 1, -1, 1, \dots$$

2. Let a = 2 and r = 5. Then:

$$\{c_n\} = c_0, c_1, c_2, c_3, c_4, \dots = 2, 10, 50, 250, 1250, \dots$$

3. Let a = 6 and r = 1/3. Then:

$$\{d_n\} = d_0, d_1, d_2, d_3, d_4, \dots = 6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \dots$$

Arithmetic Progression

Definition: An *arithmetic progression* is a sequence of the form: $a, a + d, a + 2d, \ldots, a + nd, \ldots$ where the *initial term a* and the *common difference d* are real numbers.

Ex:

1. Let
$$a = -1$$
 and $d = 4$:
 $\{s_n\} = s_0, s_1, s_2, s_3, s_4, \dots = -1, 3, 7, 11, 15, \dots$

2. Let
$$a = 7$$
 and $d = -3$:
 $\{t_n\} = t_0, t_1, t_2, t_3, t_4, \dots = 7, 4, 1, -2, -5, \dots$

3. Let
$$a = 1$$
 and $d = 2$:
 $\{u_n\} = u_0, u_1, u_2, u_3, u_4, \dots = 1, 3, 5, 7, 9, \dots$

Strings

Definition: A *string* is a finite sequence of characters from a finite set (an alphabet).

- Sequences of characters or bits are important in computer science.
- The *empty string* is represented by λ .
- The string *abcde* has *length* 5.

Recurrence Relations

- **Definition:** A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_o, a_1, ..., a_{n-1}$, for all integers n with $n \ge n_o$, where n_o is a nonnegative integer.
- The *initial conditions* for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.

Questions about Recurrence Relations

Example 1: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for n = 1,2,3,4,... and suppose that $a_0 = 2$. What are a_1 , a_2 and a_3 ? [Here $a_0 = 2$ is the initial condition.]

What are *a*₁, *a*₂, and *a*₃?

Solution: We see from the recurrence relation that

$$a_1 = a_0 + 3 = 2 + 3 = 5$$

 $a_2 = 5 + 3 = 8$
 $a_3 = 8 + 3 = 11$

Questions about Recurrence Relations

Example 2: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for n = 2,3,4,... and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3 ? [Here the initial conditions are $a_0 = 3$ and $a_1 = 5$.] What are a_2 and a_3 ?

Solution: We see from the recurrence relation that

$$a_2 = a_1 - a_0 = 5 - 3 = 2$$

 $a_3 = a_2 - a_1 = 2 - 5 = -3$

Fibonacci Sequence

Definition: Define the *Fibonacci sequence*, f_0 , f_1 , f_2 , ..., by:

- Initial Conditions: $f_0 = 0, f_1 = 1$
- Recurrence Relation: $f_n = f_{n-1} + f_{n-2}$

Example: Find f_2, f_3, f_4, f_5 and f_6 .

Answer:

$$\begin{array}{l} f_2 = f_1 + f_0 = 1 + 0 = 1, \\ f_3 = f_2 + f_1 = 1 + 1 = 2, \\ f_4 = f_3 + f_2 = 2 + 1 = 3, \\ f_5 = f_4 + f_3 = 3 + 2 = 5, \\ f_6 = f_5 + f_4 = 5 + 3 = 8. \end{array}$$

Solving Recurrence Relations

- Finding a formula for the *n*th term of the sequence generated by a recurrence relation is called *solving the recurrence relation*.
- Such a formula is called a *closed formula*.
- Many methods for solving recurrence relations (Ch. 8)
- Here we illustrate by example the method of *iteration* in which we need to guess the formula. The guess can be proved correct by the method of induction (Ch. 5).

Iterative Solution Example

Method 1: Working upward, forward substitution Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for n = 2,3,4,... and suppose that $a_1 = 2$. $a_2 = 2 + 3$ $a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2$ $a_4 = (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3$.

$$a_n = a_{n-1} + 3 = (2 + 3 \cdot (n - 2)) + 3 = 2 + 3(n - 1)$$

Iterative Solution Example

Method 2: Working downward, backward substitution Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for n = 2,3,4,... and suppose that $a_1 = 2$.

Financial Application

Example: Suppose that a person deposits \$10,000.00 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

Let P_n denote the amount in the account after 30 years. P_n satisfies the following recurrence relation:

 $P_n = P_{n-1} + 0.11P_{n-1} = (1.11) P_{n-1}$ with the initial condition $P_0 = 10,000$

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Financial Application

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11) P_{n-1}$$

with the initial condition $P_0 = 10,000$
Solution: Forward Substitution

$$\begin{split} P_{1} &= (1.11)P_{0} \\ P_{2} &= (1.11)P_{1} = (1.11)^{2}P_{0} \\ P_{3} &= (1.11)P_{2} = (1.11)^{3}P_{0} \\ &\vdots \\ P_{n} &= (1.11)P_{n-1} = (1.11)^{n}P_{0} = (1.11)^{n} \ 10,000 \\ P_{n} &= (1.11)^{n} \ 10,000 \quad \text{(Can prove by induction, covered in Ch. 5)} \\ P_{30} &= (1.11)^{30} \ 10,000 = \$228,992.97 \end{split}$$

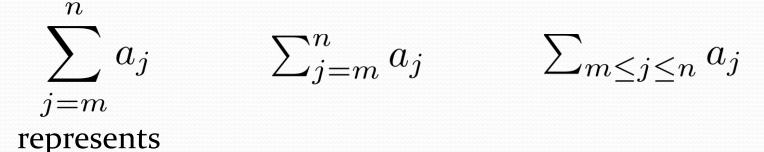
Useful Sequences

TABLE 1 Some Useful Sequences.		
nth Term	First 10 Terms	
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100,	
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000,	
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000,	
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024,	
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049,	
<i>n</i> !	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800,	
fn	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89,	

Example: Conjecture a formula for a_n if the first 10 terms of the sequence $\{a_n\}$ are 1, 7, 25, 79, 241, 727, 2185, 6559, 19681, 59047 **Solution**: $a_n = 3^n - 2$

Summations

- Sum of terms $a_m, a_{m+1}, \ldots, a_n$ from the sequence $\{a_n\}$
- The notation:

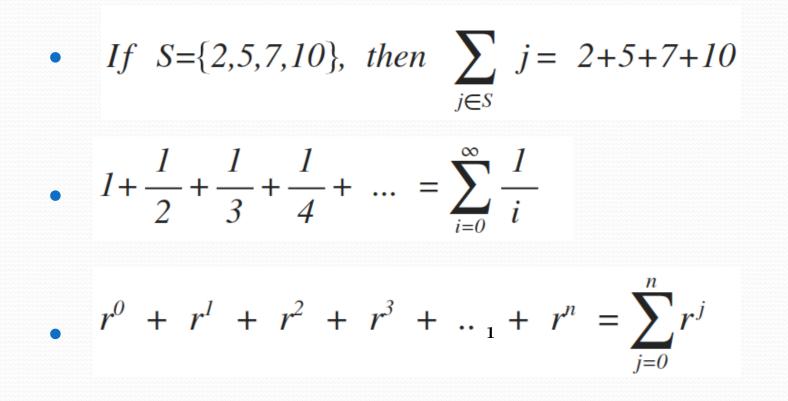


$$a_m + a_{m+1} + \dots + a_n$$

• The variable *j* is called the *index of summation*. It runs through all the integers starting with its *lower limit m* and ending with its *upper limit n*.

Summations

Examples:



Geometric Series

Sums of terms of geometric progressions

$$\sum_{j=0}^{n} ar^{j} = \begin{cases} \frac{ar^{n+1}-a}{r-1} & r \neq 1\\ (n+1)a & r = 1 \end{cases}$$

Proof:

Let
$$S_n = \sum_{j=0}^n ar^j$$

 $rS_n = r \sum_{j=0}^n ar^j$
 $= \sum_{j=0}^n ar^{j+1}$

To compute S_n , first multiply both sides of the equality by r and then manipulate the resulting sum as follows:

By the distributive property

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Geometric Series

$$= \sum_{j=0}^{n} ar^{j+1}$$
 From previous slide.
$$= \sum_{k=1}^{n+1} ar^{k}$$
 Shifting the index of summation with $k = j+1$

$$=\left(\sum_{k=0}^{n} ar^{k}\right) + (ar^{n+1} - a)$$

Removing k = n + 1 term and adding k = 0 term.

$$=S_n + (ar^{n+1} - a)$$

Substituting *S* for summation formula

$$\therefore rS_n = S_n + (ar^{n+1} - a)$$

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Geometric Series

$$rS_n = S_n + (ar^{n+1} - a)$$

$$rS_n - S_n = (ar^{n+1} - a)$$

$$S_n(r-1) = (ar^{n+1} - a)$$

From previous slide.

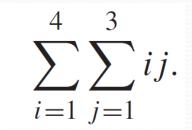
Solving for S_n

if
$$r \neq 1$$
 $S_n = \frac{ar^{n+1} - a}{r-1}$

if
$$r = 1$$
 $S_n = \sum_{j=0}^n ar^j = \sum_{j=0}^n a = (n+1)a$

QED

Double summations



• To evaluate the double sum, first expand the inner summation and then continue by computing the outer summation:

$$\sum_{i=1}^{4} \sum_{j=1}^{3} ij = \sum_{i=1}^{4} (i+2i+3i)$$
$$= \sum_{i=1}^{4} 6i$$
$$= 6 + 12 + 18 + 24 = 60.$$

Summation with set and function

- We can use summation notation to add all values of a
 - Function
 - terms of an indexed set

where the index of summation runs over all values in a set.

$$\sum_{s \in S} f(s)$$

to represent the sum of the values f(s), for all members s of S.

Example

What is the value of $\sum_{s \in \{0,2,4\}} s$?

Solution: Because $\sum_{s \in \{0,2,4\}} s$ represents the sum of the values of *s* for all the members of the set $\{0, 2, 4\}$, it follows that

$$\sum_{s \in \{0,2,4\}} s = 0 + 2 + 4 = 6.$$

Some Useful Summation Formulae

TABLE 2 Some Useful Summation Formulae.		
Sum	Closed Form	Coornetrie Series Me
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1}-a}{r-1}, r \neq 1$	Geometric Series: We just proved this.
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$	Later we will prove
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$	<pre>some of these by</pre>
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$	<induction.< td=""></induction.<>
$\sum_{k=0}^{\infty} x^k, x < 1$	$\frac{1}{1-x}$	Proof in text
$\sum_{k=1}^{\infty} kx^{k-1}, x < 1$	$\frac{1}{(1-x)^2}$	(requires calculus)