The Basics of Counting

Section 6.1

Section Summary

- Counting Rules
  - The Product Rule
  - The Sum Rule
  - The Subtraction Rule
- Examples, examples, and examples!
- Tree Diagrams

Basic Counting Principles: The Product Rule

The Product Rule : Suppose that a procedure can be broken down into a sequence of two tasks. If there are ***x*** ways to do the first task and for each of these ways of doing the first task, there are ***y*** ways to do the second task, then there are ***x.y*** ways to do the procedure.

Ex: How many bit strings of length seven are there?

Solution: Since each of the seven bits is either a 0 or a 1, the answer is $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^7 = 128$.

The Product Rule

Ex: A new company with only two employees rents a floor of a building with 12 offices. How many ways are there to assign different offices to the two employees?

Solution: There are $12 \cdot 11 = 132$ ways to assign offices.

The Product Rule

Ex: The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

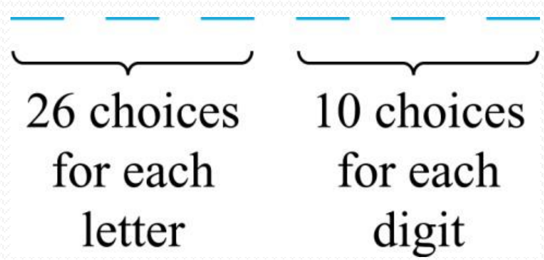
Solution: The procedure of labeling a chair consists of two tasks, namely, assigning to the seat one of the 26 uppercase English letters, and then assigning to it one of the 100 possible integers. The product rule shows that there are $26 \cdot 100 = 2600$ different ways that a chair can be labeled. Therefore, the largest number of chairs that can be labeled differently is 2600. ◀

The Product Rule

Ex: How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?

Solution: By the product rule,

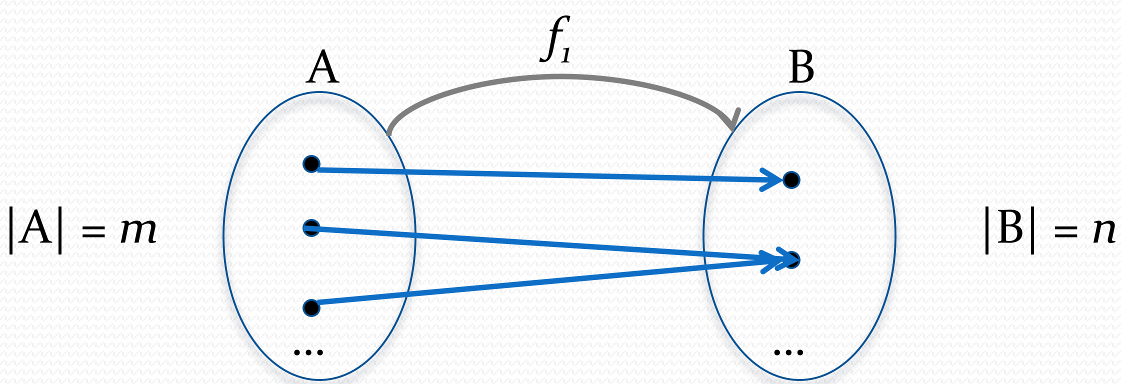
there are $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17{,}576{,}000$ different possible license plates.



Counting Functions

Counting Functions: How many functions are there from a set with m elements to a set with n elements?

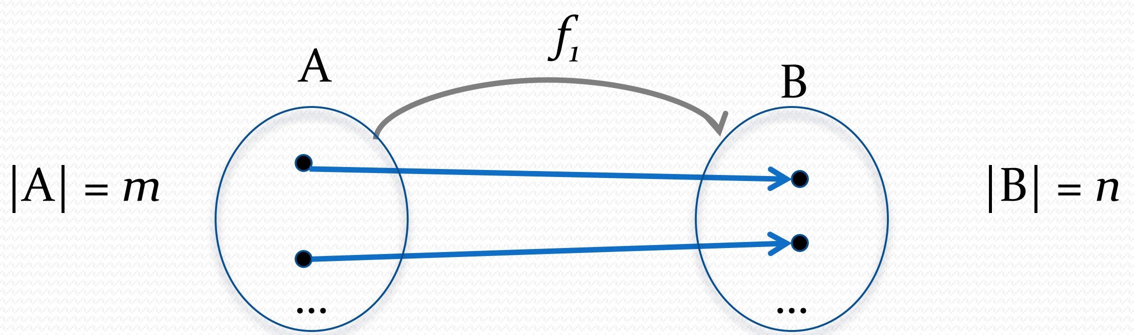
Solution: Since a function represents a choice of one of the n elements of the codomain for each of the m elements in the domain, the product rule tells us that there are $n \cdot n \cdots\ n = n^m$ such functions.



Counting Functions

Counting One-to-One Functions: How many one-to-one functions are there from a set with m elements to one with n elements?

Solution: Let $m \leq n$. Suppose the elements in the domain are a_1, a_2,…, a_m. There are n ways to choose the value of a_1 and $n-1$ ways to choose a_2, etc. The product rule tells us that there are $n(n-1)\,(n-2)\cdots(n-m+1)$ such functions.



Telephone Numbering Plan

Example: The *North American numbering plan* (*NANP*) specifies that a telephone number consists of 10 digits, with some restrictions:

	area code	office code	station code
old plan	(N Y X)	N N X	– X X X X
new plan	(N X X)	N X X	– X X X X

N = [2,9]

Y = [0,1]

X = [0,9]

How many new numbers will be available with the new plan?

Solution: The old plan can have

$(8 \cdot 2 \cdot 10)\cdot(8 \cdot 8 \cdot 10)\cdot(10 \cdot 10 \cdot 10 \cdot 10) = 1{,}024{,}000{,}000$ numbers.

The new plan can have

$(8 \cdot 10 \cdot 10)\cdot(8 \cdot 10 \cdot 10)\cdot(10 \cdot 10 \cdot 10 \cdot 10) = 6{,}400{,}000{,}000$ numbers.

Counting Subsets of a Finite Set

Counting Subsets of a Finite Set: Use the product rule to show that the number of different subsets of a finite set S is $2^{|S|}$.

Solution: When the elements of S are listed, there is a one-to-one correspondence between subsets of S and bit strings of length $|S|$. When the ith element is in the subset, the bit string has a 1, and a 0 otherwise.

By the product rule, there are $2^{|S|}$ such bit strings, and therefore $2^{|S|}$ subsets.

Product Rule in Terms of Sets

- If $A_1, A_2, \ldots, A_m$ are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements of each set.

- The task of choosing an element in the Cartesian product $A_1 \times A_2 \times \cdots \times A_m$ is done by choosing an element in A_1, an element in A_2 , …, and an element in A_m.

- By the product rule,

$$|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdot \ \ldots \ \cdot |A_m|.$$

Basic Counting Principles: The Sum Rule

The Sum Rule: If a task can be done either in one of x ways or in one of y ways, where none of the set of x ways is the same as any of the y ways, then there are $x + y$ ways to do the task.

Example: The math department must choose either a student or a faculty member as a representative for a university committee. How many choices are there if there are 37 faculty members and 83 math majors, and no one is both a faculty member and a student?

Solution: There are 37 + 83 = 120 possible ways to pick a representative.

The Sum Rule

- We can extend the sum rule to more than two tasks:

Suppose that a task can be done in one of $\boldsymbol{n_1}$ ways, in one of n_2 ways, . . . , or in one of n_m ways, where none of the set of n_i ways of doing the task is the same as any of the set of n_j ways, for all pairs *i* and *j* with $\boldsymbol{1 \le i < j \le m}$. Then the number of ways to do the task is:

$$n_1 + n_2 + \cdots n_m$$

The Sum Rule

A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

Solution: The student can choose a project by selecting a project from the first list, the second list, or the third list. Because no project is on more than one list, by the sum rule there are $23 + 15 + 19 = 57$ ways to choose a project. ◀

The Sum Rule

What is the value of k after the following code, where $n_1, n_2, \ldots, n_m$ are positive integers, has been executed?

$$
\begin{array}{l}
k := 0 \\
\textbf{for } i_1 := 1 \textbf{ to } n_1 \\
\quad\quad k := k + 1 \\
\textbf{for } i_2 := 1 \textbf{ to } n_2 \\
\quad\quad k := k + 1 \\
\quad\quad\quad\quad . \\
\quad\quad\quad\quad . \\
\quad\quad\quad\quad . \\
\textbf{for } i_m := 1 \textbf{ to } n_m \\
\quad\quad k := k + 1
\end{array}
$$

Solution: Since the initial vale of k is zero, the code is made up of m different for loops, and there n_i ways to traverse the ith loop, the value of k is $n_1 + n_2 + \cdots n_m$

The Sum Rule in Terms of Sets

- The sum rule can be phrased in terms of sets.

  $|A \cup B| = |A| + |B|$ as long as A and B are disjoint sets.
- Or more generally,

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$$
$$\text{when } A_i \cap A_j = \emptyset \text{ for all } i, j.$$

- The case where the sets have elements in common will be discussed when we consider the subtraction rule (more details in Ch. 8).

Combining the Sum and Product Rule

Example: Suppose statement labels in a programming language can be either a single lowercase letter or a single lowercase letter followed by a digit. Find the number of possible labels.

Solution: Use the product and sum rules.

$$26 + (26 \cdot 10) = 286$$

Counting Passwords

Example: Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Solution: Let P be the total number of passwords, and let P_6, P_7, and P_8 be the passwords of length 6, 7, and 8.

- By the sum rule $P = P_6 + P_7 + P_8$.

$P_6 = 36^6 - 26^6 = 2{,}176{,}782{,}336 - 308{,}915{,}776 = 1{,}867{,}866{,}560.$
$P_7 = 36^7 - 26^7 = 78{,}364{,}164{,}096 - 8{,}031{,}810{,}176 = 70{,}332{,}353{,}920.$
$P_8 = 36^8 - 26^8 = 2{,}821{,}109{,}907{,}456 - 208{,}827{,}064{,}576$
$= 2{,}612{,}282{,}842{,}880.$

$P = P_6 + P_7 + P_8 = 2{,}684{,}483{,}063{,}360.$

Basic Counting Principles: Subtraction Rule

Subtraction Rule: If a task can be done either in one of ***x*** ways or in one of ***y*** ways, then the total number of ways to do the task is ***x* + *y*** minus the number of ways to do the task that are common to the two different ways.

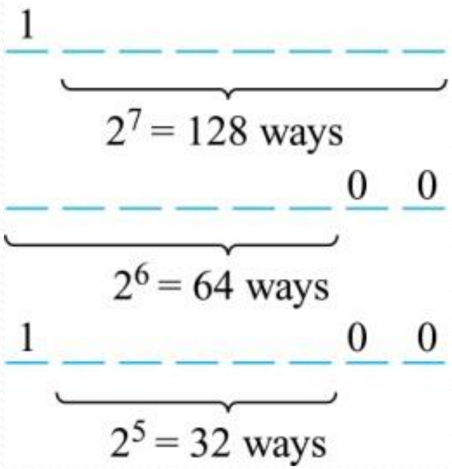
- Also known as, the *principle of inclusion-exclusion*:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Counting Bit Strings

Example: How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

Solution: Use the subtraction rule.



- Number of bit strings of length eight that start with a 1 bit: $2^7 = 128$
- Number of bit strings of length eight that end with bits 00: $2^6 = 64$
- Number of bit strings of length eight that start with a 1 bit and end with bits 00 : $2^5 = 32$

Hence, the number is $128 + 64 - 32 = 160$.

Tree Diagrams

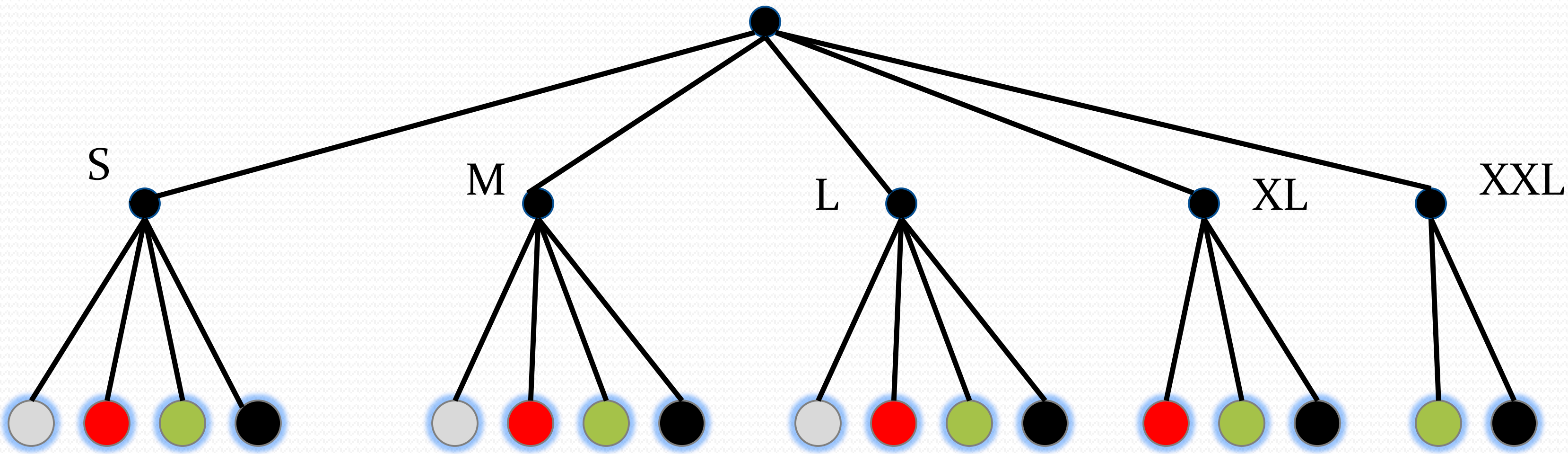
Tree Diagrams: We can solve many counting problems through the use of *tree diagrams*, where a branch represents a possible choice and the leaves represent possible outcomes.

Example: Suppose that "I ♥ Discrete Math" T-shirts come in five different sizes: S,M,L,XL, and XXL. Each size comes in four colors (white, red, green, and black), except XL, which comes only in red, green, and black, and XXL, which comes only in green and black. What is the minimum number of shirts that the campus book store needs to stock to have one of each size and color available?

Solution: Draw the tree diagram.

continued →

Tree Diagrams



- The store must stock 17 T-shirts.