
MATH 11008: Fairness Criteria

Review of the Four Fairness Criteria

- **Majority Criterion:** If candidate X has a majority of the first-place votes, then candidate X should be the winner of the election.
 - The majority criterion is satisfied by the Plurality Method, the Plurality with Elimination Method, and Pairwise Comparison Method.
 - The Borda Count Method does not satisfy the majority criterion. This means that the Borda Count Method does not *always* select the candidate with the majority of first place rankings.
- **Condorcet Criterion:** If candidate X is preferred by the voters over each of the other candidates in a head-to-head comparison, then candidate X should be the winner of the election.
 - The Condorcet criterion is always satisfied by the Method of Pairwise Comparison.
 - The Borda Count Method, the Plurality with Elimination Method, and the Plurality Method might select a Condorcet candidate, but they can also fail to honor the criterion.
- **Monotonicity Criterion:** If candidate X is a winner of an election and, in a reelection, the only changes in the ballots are changes that favor X (and only X), then X should still be the winner.
 - The Plurality Method satisfies the monotonicity criterion.
 - The Borda Count Method and the Pairwise Comparison Method can fail the monotonicity criterion because while a rearranging voter is required to move winning Candidate X favorably in his or her ranking, how that voter shifts the other candidates is not restricted.
 - The Plurality with Elimination Method can also violate the monotonicity criterion.
- **Independence of Irrelevant Alternatives Criterion:** If candidate X is a winner of an election and in a recount one of the nonwinning candidates withdraws or is disqualified, then X should still be the winner.
 - The Plurality Method, the Borda Count Method, the Pairwise Comparison Method, and the Plurality with Elimination Method can fail to satisfy the Irrelevant Alternative criterion.

Summary of the four fairness criteria

	Plurality Method	Plurality with Elimination Method	Pairwise Comparison Method	Borda Count Method
Majority Criterion	satisfied	satisfied	satisfied	not satisfied
Condorcet Criterion	not satisfied	not satisfied	satisfied	not satisfied
Monotonicity Criterion	satisfied	not satisfied	not satisfied	not satisfied
Irrelevant Alternative Criterion	not satisfied	not satisfied	not satisfied	not satisfied

- In 1952, mathematical economist **Kenneth Arrow** discovered a remarkable fact, now referred to as **Arrow's Impossibility Theorem**.

Arrow's Impossibility Theorem: It is mathematically impossible for a democratic voting method to satisfy all four of the fairness criteria.

- Maybe his conditions are too strict? Since Arrow's original work, many investigators have developed a wide array of desirable fairness rules and showed results similar to Arrow's theorem.
- In 1972 Arrow was awarded the Nobel Prize in Economics (there is no Nobel Prize in Mathematics) for his pioneering work in what is now known as *social-choice theory*, a discipline that combines aspects of mathematics, economics, and political science.

Example 1: Consider the election given by the following preference schedule:

Number of voters	10	6	5	4	2
1st choice	A	B	C	D	D
2nd choice	C	D	B	C	A
3rd choice	D	C	D	B	B
4th choice	B	A	A	A	C

- (a) Find the Condorcet candidate in this election.
- (b) Use the plurality with elimination method to find the winner of the election.
- (c) Suppose D drops out of the race. Use the plurality with elimination method to find the winner of the election after D is removed from the preference schedule.
- (d) The results of (a), (b), and (c) show that the plurality with elimination method violates several of the fairness criterion. Which ones? Explain.

Example 2: A 17 member committee is selecting a site for the next meeting. The choices are Dallas (D), Chicago (C), Atlanta (A), and Boston (B).

- (a) The committee decides to use the plurality with elimination method to select a site in a nonbinding decision. Prior to any discussion, the 17 members rank the choices according to the following voter profile:

Number of voters	Ranking
6	$A > C > B > D$
5	$B > D > A > C$
4	$C > D > B > A$
2	$C > A > B > D$

Determine which city is selected by the plurality with elimination method in the preliminary nonbinding decision.

- (b) The 2 committee members with the bottom rankings rearrange their ranking after listening to discussions. The other 15 committee members stick with their original rankings. For the official vote, the 17 members rank the choices according to the following voter profile:

Number of voters	Ranking
6	$A > C > B > D$
5	$B > D > A > C$
4	$C > D > B > A$
2	$A > C > B > D$

Use the plurality with elimination Method to determine the site selection of the committee.

- (c) According to the results of (a) and (b), the plurality with elimination method violates which fairness criterion in this selection process? Justify your answer.

Example 3: Thirty voters rank five alternatives a, b, c, d , and e according to the following voter profile:

Number of voters	16	3	6	2	3
1st choice	b	a	c	d	e
2nd choice	e	c	a	a	c
3rd choice	c	b	e	c	a
4th choice	d	d	b	e	d
5th choice	a	e	d	b	b

- (a) Which alternative has a majority of first-place votes?
- (b) Use the Borda count Method to determine which alternative the voters prefer.
- (c) According to the results of (a) and (b), the Borda count method violates which fairness criterion in this selection process? Justify your answer.

Example 4: A senator is holding a workshop called “Getting Our Priorities Straight.” To begin the discussion, the senator asked the 21 workshop members to rank the issues of job creation (j), education (e), health care (h), and gun control (g) in the order of importance to themselves and the counties they represent. The workshop members rank the issues according to the following voter profile:

Number of voters	6	5	4	3	3
1st choice	h	e	g	j	e
2nd choice	j	g	j	h	j
3rd choice	g	j	h	g	h
4th choice	e	h	e	e	g

- (a) Which issue is a Condorcet candidate?
- (b) Use the plurality method to determine which issue the workshop members felt was the highest priority.
- (c) Use the plurality with elimination to determine which issue the workshop members felt was the highest priority.
- (d) Use the Borda count method to determine which issue the workshop members felt was the highest priority.
- (e) Which voting methods – plurality, plurality with elimination, or Borda count – violate(s) the Condorcet criterion for this profile of voters? Justify your answer.