Section 2.1: Limits Graphically

Definition. We say that the limit of \( f(x) \) as \( x \) approaches \( a \) is equal to \( L \), written
\[
\lim_{x \to a} f(x) = L,
\]
if we can make the values of \( f(x) \) as close to \( L \) as we like by taking \( x \) to be sufficiently close to \( a \), but not equal to \( a \). In other words, as \( x \) approaches \( a \) (but never equaling \( a \)), \( f(x) \) approaches \( L \).

Definition. We say that the limit of \( f(x) \) as \( x \) approaches \( a \) from the left is equal to \( L \), written
\[
\lim_{x \to a^-} f(x) = L,
\]
if we can make the values of \( f(x) \) as close to \( L \) as we like by taking \( x \) to be sufficiently close to \( a \), but strictly less than \( a \) (i.e., to the left of \( a \) as viewed on a number line). In other words, as \( x \) approaches \( a \) from the left (i.e., \( x < a \)), \( f(x) \) approaches \( L \).

Definition. We say that the limit of \( f(x) \) as \( x \) approaches \( a \) from the right is equal to \( L \), written
\[
\lim_{x \to a^+} f(x) = L,
\]
if we can make the values of \( f(x) \) as close to \( L \) as we like by taking \( x \) to be sufficiently close to \( a \), but strictly greater than \( a \) (i.e., to the right of \( a \) as viewed on a number line). In other words, as \( x \) approaches \( a \) from the right (i.e., \( a < x \)), \( f(x) \) approaches \( L \).

Definition. Limits taken from the left or the right are called one-sided limits.

Result. If both one-sided limits equal \( L \), then the two-sided limit must also equal \( L \). Conversely, if the two-sided limit equals \( L \), then both one-sided limits must also equal \( L \). That is,
\[
\lim_{x \to a^-} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^+} f(x) = L \quad \text{and} \quad \lim_{x \to a} f(x) = L.
\]

Definition. The function \( f \) is continuous at \( x = a \) provided \( f(a) \) is defined, \( \lim_{x \to a} f(x) \) exists, and
\[
\lim_{x \to a} f(x) = f(a).
\]
In other words, the value of the limit equals the value of the function. Graphically, the function \( f \) is continuous at \( x = a \) provided the graph of \( y = f(x) \) does not have any holes, jumps, or breaks at \( x = a \). (That is, the function is connected at \( x = a \).)

If \( f \) is not continuous at \( x = a \), then we say \( f \) is discontinuous at \( x = a \) (or \( f \) has a discontinuity at \( x = a \)).
Example 1. For the function $f$ graphed below, find the following:

1. $\lim_{x \to -3^-} f(x) = \phantom{-}$
2. $\lim_{x \to -3^+} f(x) = \phantom{-}$
3. $\lim_{x \to -3} f(x) = \phantom{-}$
4. $f(-3) = \phantom{-}$
5. $\lim_{x \to -1^-} f(x) = \phantom{-}$
6. $\lim_{x \to -1^+} f(x) = \phantom{-}$
7. $\lim_{x \to -1} f(x) = \phantom{-}$
8. $f(-1) = \phantom{-}$
9. $\lim_{x \to 2^-} f(x) = \phantom{-}$
10. $\lim_{x \to 2^+} f(x) = \phantom{-}$
11. $\lim_{x \to 2} f(x) = \phantom{-}$
12. $f(2) = \phantom{-}$
13. $\lim_{x \to 4^-} f(x) = \phantom{-}$
14. $\lim_{x \to 4^+} f(x) = \phantom{-}$
15. $\lim_{x \to 4} f(x) = \phantom{-}$
16. $f(4) = \phantom{-}$
17. $\lim_{x \to 6^-} f(x) = \phantom{-}$
18. $\lim_{x \to 6^+} f(x) = \phantom{-}$
19. $\lim_{x \to 6} f(x) = \phantom{-}$
20. $f(6) = \phantom{-}$
21. List the value(s) of $x$ at which $f$ is discontinuous.

Note that the function is continuous at $x = 4$ and hence

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^+} f(x) = \lim_{x \to 4} f(x) = f(4) = -1.$$
EXERCISES

For the function $f$ graphed below, find the following:

1. \( \lim_{x \to -5^-} f(x) = \)
2. \( \lim_{x \to -5^+} f(x) = \)
3. \( \lim_{x \to 5} f(x) = \)
4. \( f(-5) = \)
5. \( \lim_{x \to -2^-} f(x) = \)
6. \( \lim_{x \to -2^+} f(x) = \)
7. \( \lim_{x \to 2} f(x) = \)
8. \( f(-2) = \)
9. \( \lim_{x \to 0^-} f(x) = \)
10. \( \lim_{x \to 0^+} f(x) = \)
11. \( \lim_{x \to 0} f(x) = \)
12. \( f(0) = \)
13. \( \lim_{x \to 1^-} f(x) = \)
14. \( \lim_{x \to 1^+} f(x) = \)
15. \( \lim_{x \to 1} f(x) = \)
16. \( f(1) = \)
17. \( \lim_{x \to 3^-} f(x) = \)
18. \( \lim_{x \to 3^+} f(x) = \)
19. \( \lim_{x \to 3} f(x) = \)
20. \( f(3) = \)
21. \( \lim_{x \to 4^-} f(x) = \)
22. \( \lim_{x \to 4^+} f(x) = \)
23. \( \lim_{x \to 4} f(x) = \)
24. \( f(4) = \)
25. \( \lim_{x \to 5^-} f(x) = \)
26. \( \lim_{x \to 5^+} f(x) = \)
27. \( \lim_{x \to 5} f(x) = \)
28. \( f(5) = \)
29. List the value(s) of $x$ at which $f$ is discontinuous.
ANSWERS

1. 2  
2. 2  
3. 2  
4. 1  
5. −1  
6. 4  
7. Does not exist  
8. 2  
9. 6  
10. 6  
11. 6  
12. 6  
13. 7  
14. 4  
15. Does not exist  
16. 7  
17. 2  
18. 2  
19. 2  
20. Undefined  
21. 1  
22. −4  
23. Does not exist  
24. 1  
25. −3  
26. 1  
27. Does not exist  
28. 1  
29. $x = -5, -2, 1, 3, 4, 5$