Chapter 12

4-vectors

We now come to a very powerful concept in relativity, namely that of *4-vectors*. Although it is possible to derive everything in special relativity without the use of 4-vectors (and indeed, this is the route, give or take, that we took in the previous two chapters), they are *extremely* helpful in making calculations and concepts much simpler and more transparent.

I have chosen to postpone the introduction to 4-vectors until now, in order to make it clear that everything in special relativity can be derived without them. In encountering relativity for the first time, it's nice to know that no "advanced" techniques are required. But now that you've seen everything once, let's go back and derive various things in an easier way.

This situation, where 4-vectors are helpful but not necessary, is more pronounced in general relativity, where the concept of *tensors* (the generalization of 4-vectors) is, for all practical purposes, completely necessary for an understanding of the subject. We won't have time to go very deeply into GR in Chapter 13, so you'll have to just accept this fact. But suffice it to say that an eventual understanding of GR requires a firm understanding of special-relativity 4-vectors.

12.1 Definition of 4-vectors

Definition 12.1 *The 4-tuplet, $A = (A_0, A_1, A_2, A_3)$, is a "4-vector" if the A_i transform under a Lorentz transformation in the same way that $(c\,dt, dx, dy, dz)$ do. In other words, they must transform like (assuming the LT is along the x-direction; see Fig. 12.1):*

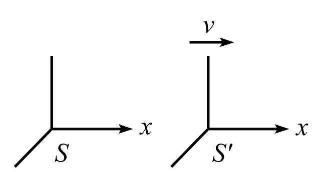

Figure 12.1

$$\begin{aligned} A_0 &= \gamma(A'_0 + (v/c)A'_1), \\ A_1 &= \gamma(A'_1 + (v/c)A'_0), \\ A_2 &= A'_2, \\ A_3 &= A'_3. \end{aligned} \tag{12.1}$$

Remarks:

1. Similar equations must hold, of course, for Lorentz transformations in the y- and z-directions.

2. Additionally, the last three components must be a vector in 3-space. That is, they must transform like a usual vector under rotations in 3-space.

3. We'll use a capital Roman letter to denote a 4-vector. A bold-face letter will denote, as usual, a vector in 3-space.

4. Lest we get tired of writing the c's over and over, we will henceforth work in units where $c = 1$.

5. The first component of a 4-vector is called the "time" component. The other three are the "space" components.

6. The components in (dt, dx, dy, dz) are sometimes referred to as (dx_0, dx_1, dx_2, dx_3). Also, some treatments use the indices "1" through "4", with "4" being the "time" component. But we'll use "0" through "3".

7. The A_i may be functions of the dx_i, the x_i and their derivatives, any invariants (that is, frame-independent quantities) such as the mass m, and v.

8. 4-vectors are the obvious generalization of vectors in regular space. A vector in 3-dimensions, after all, is something that transforms under a rotation just like (dx, dy, dz) does. We have simply generalized a 3-D rotation to a 4-D Lorentz transformation. ♣

12.2 Examples of 4-vectors

So far, we have only one 4-vector at our disposal, namely (dt, dx, dy, dz). What are some others? Well, $(7dt, 7dx, 7dy, 7dz)$ certainly works, as does any other constant multiple of (dt, dx, dy, dz). Indeed, $m(dt, dx, dy, dz)$ is a 4-vector, because m is an invariant (independent of frame).

How about $A = (dt, 2dx, dy, dz)$? No, this isn't a 4-vector, because on one hand it must transform (assuming it's a 4-vector) like

$$\begin{aligned}
dt \equiv A_0 &= \gamma(A'_0 + vA'_1) \equiv \gamma\Big(dt' + v(2\,dx')\Big),\\
2\,dx \equiv A_1 &= \gamma(A'_1 + vA'_0) \equiv \gamma\Big((2\,dx') + v\,dt'\Big),\\
dy \equiv A_2 &= A'_2 \equiv dy',\\
dz \equiv A_3 &= A'_3 \equiv dz',
\end{aligned} \tag{12.2}$$

from the definition of a 4-vector. But on the other hand, it transforms like

$$\begin{aligned}
dt &= \gamma(dt' + v\,dx'),\\
2\,dx &= 2\gamma(dx' + v\,dt'),\\
dy &= dy',\\
dz &= dz',
\end{aligned} \tag{12.3}$$

because this is how the dx_i transform. The two preceding sets of equations are inconsistent, so $A = (dt, 2dx, dy, dz)$ is not a 4-vector. Note that if we had instead considered the 4-tuplet, $A = (dt, dx, 2dy, dz)$, then the two preceding equations would have been consistent. But if we had then looked at how A transforms under a Lorentz transformation in the y-direction, we would have found that it is not a 4-vector.

The moral of this story is that the above definition of a 4-vector is a nontrivial one because there are two possible ways that a 4-tuplet can transform. It can transform according to the 4-vector definition, as in eq. (12.2). Or, it can transform by simply having each of the A_i transform separately (knowing how the dx_i transform), as in eq. (12.3). Only for certain special 4-tuplets do these two methods give the same result. By definition, we label these special 4-tuplets as 4-vectors.

Let us now construct some less trivial examples of 4-vectors. In constructing these, we will make abundant use of the fact that the proper-time interval, $d\tau \equiv \sqrt{dt^2 - d\mathbf{r}^2}$, is an invariant.

- **Velocity 4-vector:** We can divide (dt, dx, dy, dz) by $d\tau$, where $d\tau$ is the proper time between two events (the same two events that yielded the dt, etc.). The result is indeed a 4-vector, because $d\tau$ is independent of the frame in which it is measured. Using $d\tau = dt/\gamma$, we obtain

$$V \equiv \frac{1}{d\tau}(dt, dx, dy, dz) = \gamma\left(1, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = (\gamma, \gamma\mathbf{v}) \tag{12.4}$$

  is a 4-vector. This is known as the *velocity 4-vector*. In the rest frame of the object we have $\mathbf{v} = \mathbf{0}$, so V reduces to $V = (1, 0, 0, 0)$. With the c's, we have $V = (\gamma c, \gamma\mathbf{v})$.

- **Energy-momentum 4-vector:** If we multiply the velocity 4-vector by the invariant m, we obtain another 4-vector,

$$P \equiv mV = (\gamma m, \gamma m\mathbf{v}) = (E, \mathbf{p}), \tag{12.5}$$

  which is known as the *energy-momentum 4-vector* (or the *4-momentum* for short), for obvious reasons. In the rest frame of the object, P reduces to $P = (m, 0, 0, 0)$. With the c's, we have $P = (\gamma mc, \gamma m\mathbf{v}) = (E/c, \mathbf{p})$. Some treatments multiply through by c, so that the 4-momentum is $(E, \mathbf{p}c)$.

- **Acceleration 4-vector:** We can also take the derivative of the velocity 4-vector with respect to τ. The result is indeed a 4-vector, because taking the derivative simply entails taking the difference between two 4-vectors (which results in a 4-vector because eq. (12.1) is linear), and then dividing by the invariant $d\tau$ (which again results in a 4-vector). Using $d\tau = dt/\gamma$, we obtain

$$A \equiv \frac{dV}{d\tau} = \frac{d}{d\tau}(\gamma, \gamma\mathbf{v}) = \gamma\left(\frac{d\gamma}{dt}, \frac{d(\gamma\mathbf{v})}{dt}\right). \tag{12.6}$$

  Using $d\gamma/dt = v\dot{v}/(1-v^2)^{3/2} = \gamma^3 v\dot{v}$, we have

$$A = (\gamma^4 v\dot{v}\,,\, \gamma^4 v\dot{v}\mathbf{v} + \gamma^2\mathbf{a}), \tag{12.7}$$

  where $\mathbf{a} \equiv d\mathbf{v}/dt$. A is known as the *acceleration 4-vector*. In the rest frame of the object (or, rather, in the instantaneous inertial frame), A reduces to $A = (0, \mathbf{a})$.

As we always do, we will pick the relative velocity, $\mathbf{v}$, to point in the x-direction. That is, $\mathbf{v} = (v_x, 0, 0)$. This means that $v = v_x$, and also that $\dot{v} = \dot{v}_x \equiv a_x$.[1] Eq. (12.7) then becomes

$$\begin{aligned} A &= (\gamma^4 v_x a_x\,,\, \gamma^4 v_x^2 a_x + \gamma^2 a_x\,,\, \gamma^2 a_y\,,\, \gamma^2 a_z) \\ &= (\gamma^4 v_x a_x\,,\, \gamma^4 a_x\,,\, \gamma^2 a_y\,,\, \gamma^2 a_z). \end{aligned} \tag{12.8}$$

We can keep taking derivatives with respect to τ to create other 4-vectors, but these have little relevance in the real world.

- **Force 4-vector:** We define the *force 4-vector* as

$$F \equiv \frac{dP}{d\tau} = \gamma \left(\frac{dE}{dt}, \frac{d\mathbf{p}}{dt} \right) = \gamma \left(\frac{dE}{dt}, \mathbf{f} \right), \tag{12.9}$$

where $\mathbf{f} \equiv d(\gamma m \mathbf{v})/dt$ is the usual 3-force. We'll use $\mathbf{f}$ instead of $\mathbf{F}$ in this chapter, to avoid confusion with the 4-force, F.

In the case where m is constant,[2] F can be written as $F = d(mV)/d\tau = m\, dV/d\tau = mA$. We therefore still have a nice "F equals mA" law of physics, but it's now a 4-vector equation instead of the old 3-vector one. In terms of the acceleration 4-vector, we may use eq. (12.7) to write (if m is constant)

$$F = mA = (\gamma^4 m v \dot{v}\,,\, \gamma^4 m v \dot{v} \mathbf{v} + \gamma^2 m \mathbf{a}). \tag{12.10}$$

In the rest frame of the object (or, rather, the instantaneous inertial frame), F reduces to $F = (0, \mathbf{f})$, because $dE/dt = 0$, as you can verify. Also, mA reduces to $mA = (0, m\mathbf{a})$. Therefore, $F = mA$ reduces to the familiar $\mathbf{f} = m\mathbf{a}$.

12.3 Properties of 4-vectors

The appealing thing about 4-vectors is that they have many useful properties. Let's look at some of these.

- **Linear combinations:** If A and B are 4-vectors, then $C \equiv aA + bB$ is also a 4-vector. This is true because the transformations in eq. (12.1) are linear (as we noted above when deriving the acceleration 4-vector). This linearity implies that the transformation of, say, the time component is

$$\begin{aligned} C_0 \equiv (aA + bB)_0 = aA_0 + bB_0 &= a(A'_0 + vA'_1) + b(B'_0 + vB'_1) \\ &= (aA'_0 + bB'_0) + v(aA'_1 + bB'_1) \\ &\equiv C'_0 + vC'_1, \end{aligned} \tag{12.11}$$

which is the proper transformation for the time component of a 4-vector. Likewise for the other components. This property holds, of course, just as it does for linear combinations of vectors in 3-space.

[1] The acceleration vector, $\mathbf{a}$, is free to point in any direction, but you can check that the 0's in $\mathbf{v}$ lead to $\dot{v} = a_x$. See Exercise 1.

[2] The mass m would not be constant if the object were being heated, or if extra mass were being added to it. We won't concern ourselves with such cases here.

- **Inner-product invariance:** Consider two arbitrary 4-vectors, A and B. Define their inner product to be

$$A \cdot B \equiv A_0 B_0 - A_1 B_1 - A_2 B_2 - A_3 B_3 \equiv A_0 B_0 - \mathbf{A} \cdot \mathbf{B}. \tag{12.12}$$

Then $A \cdot B$ is invariant. That is, it is independent of the frame in which it is calculated. This can be shown by direct calculation, using the transformations in eq. (12.1):

$$\begin{aligned}
A \cdot B &\equiv A_0 B_0 - A_1 B_1 - A_2 B_2 - A_3 B_3 \\
&= \Big(\gamma(A_0' + vA_1')\Big)\Big(\gamma(B_0' + vB_1')\Big) - \Big(\gamma(A_1' + vA_0')\Big)\Big(\gamma(B_1' + vB_0')\Big) \\
&\qquad -A_2' B_2' - A_3' B_3' \\
&= \gamma^2\Big(A_0' B_0' + v(A_0' B_1' + A_1' B_0') + v^2 A_1' B_1'\Big) \\
&\quad -\gamma^2\Big(A_1' B_1' + v(A_1' B_0' + A_0' B_1') + v^2 A_0' B_0'\Big) \\
&\qquad -A_2' B_2' - A_3' B_3' \\
&= A_0' B_0'(\gamma^2 - \gamma^2 v^2) - A_1' B_1'(\gamma^2 - \gamma^2 v^2) - A_2' B_2' - A_3' B_3' \\
&= A_0' B_0' - A_1' B_1' - A_2' B_2' - A_3' B_3' \\
&\equiv A' \cdot B'.
\end{aligned} \tag{12.13}$$

The importance of this result cannot be overstated. This invariance is analogous to the invariance of the inner product, $\mathbf{A} \cdot \mathbf{B}$, for rotations in 3-space. The above inner product is also invariant under rotations in 3-space, because it involves the combination $\mathbf{A} \cdot \mathbf{B}$.

The minus signs in the inner product may seem a little strange. But the goal was to find a combination of two arbitrary vectors that is invariant under a Lorentz transformation (because such combinations are very useful in seeing what is going on in a problem). The nature of the LT's demands that there be opposite signs in the inner product, so that's the way it is.

- **Norm:** As a corollary to the invariance of the inner product, we can look at the inner product of a 4-vector with itself, which is by definition the square of the norm. We see that

$$A^2 \equiv A \cdot A \equiv A_0 A_0 - A_1 A_1 - A_2 A_2 - A_3 A_3 = A_0^2 - |\mathbf{A}|^2 \tag{12.14}$$

is invariant. This is analogous to the invariance of the norm $\sqrt{\mathbf{A} \cdot \mathbf{A}}$ for rotations in 3-space. Special cases of the invariance of the 4-vector norm are the invariance of $c^2 t^2 - x^2$ in eq. (10.37), and the invariance of $E^2 - p^2$ in eq. (11.20).

- **A theorem:** Here's a nice little theorem:

If a certain one of the components of a 4-vector is 0 in every frame, then all four components are 0 in every frame.

Proof: If one of the space components (say, A_1) is 0 in every frame, then the other space components must also be 0 in every frame, because otherwise a rotation would make $A_1 \neq 0$. Also, the time component A_0 must be 0 in every frame, because otherwise a Lorentz transformation in the x-direction would make $A_1 \neq 0$.

If the time component, A_0, is 0 in every frame, then the space components must also be 0 in every frame, because otherwise a Lorentz transformation in the appropriate direction would make $A_0 \neq 0$. ■

If someone comes along and says that she has a vector in 3-space that has no x-component, no matter how you rotate the axes, then you would certainly say that the vector must obviously be the zero vector. The situation in Lorentzian 4-space is basically the same, because all the coordinates get intertwined with each other in the Lorentz (and rotation) transformations.

12.4 Energy, momentum

12.4.1 Norm

Many useful things arise from the simple fact that the P in eq. (12.5) is a 4-vector. The invariance of the norm implies that $P \cdot P = E^2 - |\mathbf{p}|^2$ is invariant. If we are dealing with only one particle, we can determine the value of P^2 by conveniently working in the rest frame of the particle (so that $\mathbf{v} = \mathbf{0}$). We obtain

$$E^2 - p^2 = m^2, \tag{12.15}$$

or $E^2 - p^2c^2 = m^2c^4$, with the c's. We already knew this, of course, from just writing out $E^2 - p^2 = \gamma^2 m^2 - \gamma^2 m^2 v^2 = m^2$.

For a collection of particles, knowledge of the norm is very useful. If a process involves many particles, then we can say that for *any* subset of the particles,

$$\left(\sum E\right)^2 - \left(\sum \mathbf{p}\right)^2 \qquad \text{is invariant,} \tag{12.16}$$

because this is simply the norm of the sum of the energy-momentum 4-vectors of the chosen particles. The sum is again a 4-vector, due to the linearity of eqs. (12.1).

What is the value of the invariant in eq. (12.16)? The most concise description (which is basically a tautology) is that it is the square of the energy in the CM frame, that is, in the frame where $\sum \mathbf{p} = \mathbf{0}$. For one particle, this reduces to m^2.

Note that the sums are taken before squaring in eq. (12.16). Squaring before adding would simply give the sum of the squares of the masses.

12.4.2 Transformation of E,p

We already know how the energy and momentum transform (see Section 11.2), but let's derive the transformation again here in a very quick and easy manner. We know that (E, p_x, p_y, p_z) is a 4-vector. So it must transform according to eq. (12.1). Therefore (for an LT in the x-direction),

$$\begin{aligned} E &= \gamma(E' + vp'_x), \\ p_x &= \gamma(p'_x + vE'), \\ p_y &= p'_y, \\ p_z &= p'_z, \end{aligned} \tag{12.17}$$

in agreement with eq. (11.18). That's all there is to it.

REMARK: The fact that E and $\mathbf{p}$ are part of the same 4-vector provides an easy way to see that if one of them is conserved (in every frame) in a collision, then the other is also. Consider an interaction among a set of particles, and look at the 4-vector, $\Delta P \equiv P_{\text{after}} - P_{\text{before}}$. If E is conserved in every frame, then the time component of ΔP is 0 in every frame. But then the theorem in the previous section says that all four components of ΔP are 0 in every frame. Therefore, $\mathbf{p}$ is conserved. Likewise for the case where one of the p_i is known to be conserved. ♣

12.5 Force and acceleration

Throughout this section, we will deal with objects with constant mass, which we will call "particles". The treatment here can be generalized to cases where the mass changes (for example, the object is being heated, or extra mass is being dumped on it), but we won't concern ourselves with these.

12.5.1 Transformation of forces

Let's first look at the force 4-vector in the instantaneous inertial frame of a given particle (frame S'). Eq. (12.9) gives

$$F' = \gamma\left(\frac{dE'}{dt}, \mathbf{f}'\right) = (0, \mathbf{f}'). \tag{12.18}$$

The first component is zero because $dE'/dt = d(m/\sqrt{1-v'^2}\,)/dt$, and this carries a factor of v', which is zero in this frame. Equivalently, you can just use eq.(12.10), with a speed of zero.

We can now write down two expressions for the 4-force, F, in another frame, S. First, since F is a 4-vector, it transforms according to eq. (12.1). We therefore have, using eq. (12.18),

$$\begin{aligned} F_0 &= \gamma(F'_0 + vF'_1) = \gamma v f'_x, \\ F_1 &= \gamma(F'_1 + vF'_0) = \gamma f'_x, \\ F_2 &= F'_2 = f'_y, \\ F_3 &= F'_3 = f'_z. \end{aligned} \tag{12.19}$$

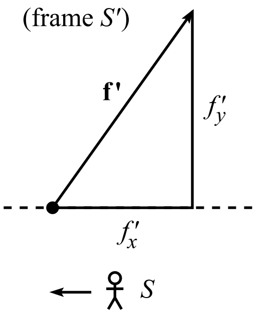

Figure 12.2

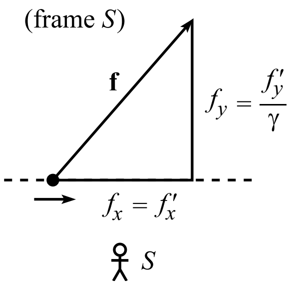

Figure 12.3

But second, from the definition in eq. (12.9), we also have

$$\begin{aligned} F_0 &= \gamma\, dE/dt, \\ F_1 &= \gamma f_x, \\ F_2 &= \gamma f_y, \\ F_3 &= \gamma f_z. \end{aligned} \tag{12.20}$$

Combining eqs. (12.19) and (12.20), we obtain

$$\begin{aligned} dE/dt &= v f'_x, \\ f_x &= f'_x, \\ f_y &= f'_y/\gamma, \\ f_z &= f'_z/\gamma. \end{aligned} \tag{12.21}$$

We therefore recover the results of Section 11.5.3. The longitudinal force is the same in both frames, but the transverse forces are larger by a factor of γ in the particle's frame. Hence, f_y/f_x decreases by a factor of γ when going from the particle's frame to the lab frame (see Fig. 12.2 and Fig. 12.3).

As a bonus, the F_0 component in eq. (12.21) tells us (after multiplying through by dt) that $dE = f_x\, dx$, which is the work-energy result. In other words, using $f_x \equiv dp_x/dt$, we have just proved again the result, $dE/dx = dp/dt$, from Section 11.5.1.

As noted in Section 11.5.3, we can't switch the S and S' frames and write $f'_y = f_y/\gamma$. When talking about the forces on a particle, there is indeed one preferred frame of reference, namely that of the particle. All frames are not equivalent here. When forming all of our 4-vectors in Section 12.2, we explicitly used the $d\tau$, dt, dx, etc., from two events, and it was understood that these two events were located at the particle.

12.5.2 Transformation of accelerations

The procedure here is similar to the above treatment of the force. Let's first look at the acceleration 4-vector in the instantaneous inertial frame of a given particle (frame S'). Eq. (12.7) or eq. (12.8) gives

$$A' = (0, \mathbf{a}'), \tag{12.22}$$

because $v' = 0$ in S'.

We can now write down two expressions for the 4-acceleration, A, in another frame, S. First, since A is a 4-vector, it transforms according to eq. (12.1). So we have, using eq. (12.22),

$$\begin{aligned} A_0 &= \gamma(A'_0 + vA'_1) = \gamma v a'_x, \\ A_1 &= \gamma(A'_1 + vA'_0) = \gamma a'_x, \\ A_2 &= A'_2 = a'_y, \\ A_3 &= A'_3 = a'_z. \end{aligned} \tag{12.23}$$

But second, from the expression in eq. (12.8), we also have

$$\begin{aligned} A_0 &= \gamma^4 v a_x, \\ A_1 &= \gamma^4 a_x, \\ A_2 &= \gamma^2 a_y, \\ A_3 &= \gamma^2 a_z. \end{aligned} \tag{12.24}$$

Combining eqs. (12.23) and (12.24), we obtain

$$\begin{aligned} a_x &= a'_x/\gamma^3, \\ a_x &= a'_x/\gamma^3, \\ a_y &= a'_y/\gamma^2, \\ a_z &= a'_z/\gamma^2. \end{aligned} \tag{12.25}$$

(The first two equations here are redundant.) We see that a_y/a_x increases by a factor of $\gamma^3/\gamma^2 = \gamma$ when going from the particle's frame to the lab frame (see Fig. 12.4 and Fig. 12.5). This is the opposite of the effect on f_y/f_x.[3] This difference makes it clear that an $\mathbf{f} = m\mathbf{a}$ law wouldn't make any sense. If it were true in one frame, then it wouldn't be true in another.

Note also that the increase in a_y/a_x in going to the lab frame is consistent with length contraction, as the Bead-on-a-rod example in Section 11.5.3 showed.

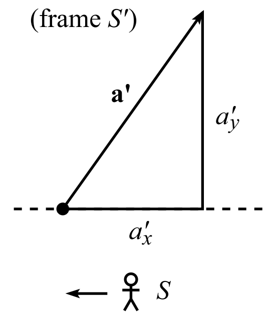

Figure 12.4

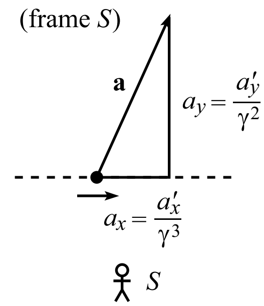

Figure 12.5

Example (Acceleration for circular motion): A particle moves with constant speed v along the circle $x^2 + y^2 = r^2$, $z = 0$, in the lab frame. At the instant the particle crosses the negative y-axis (see Fig. 12.6), find the 3-acceleration and 4-acceleration in both the lab frame and the instantaneous rest frame of the particle (with axes chosen parallel to the lab's axes).

Solution: Let the lab frame be S, and let the particle's instantaneous inertial frame be S' when it crosses the negative y-axis. Then S and S' are related by a Lorentz transformation in the x-direction.

The 3-acceleration in S is simply

$$\mathbf{a} = (0, v^2/r, 0). \tag{12.26}$$

There's nothing fancy going on here; the nonrelativistic proof of $a = v^2/r$ works just fine again in the relativistic case. Eq. (12.7) or (12.8) then gives the 4-acceleration in S as

$$A = (0, 0, \gamma^2 v^2/r, 0). \tag{12.27}$$

To find the acceleration vectors in S', we will use the fact S' and S are related by a Lorentz transformation in the x-direction. Therefore, the A_2 component of the 4-acceleration is unchanged. So the 4-acceleration in S' is also

$$A' = A = (0, 0, \gamma^2 v^2/r, 0). \tag{12.28}$$

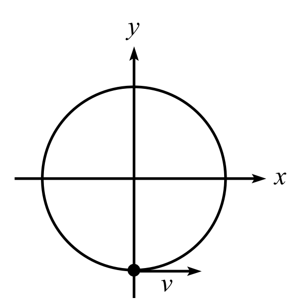

Figure 12.6

[3]In a nutshell, this difference is due to the fact that γ changes with time. When talking about accelerations, there are γ's that we have to differentiate; see eq. (12.6). This isn't the case with forces, because the γ is absorbed into the definition of $\mathbf{p} \equiv \gamma m\mathbf{v}$; see eq. (12.9). This is what leads to the different powers of γ in eq. (12.24), in contrast with the identical powers in eq. (12.20).

In the particle's frame, $\mathbf{a}'$ is simply the space part of A (using eq. (12.7) or (12.8), with $v = 0$ and $\gamma = 1$). Therefore, the 3-acceleration in S' is

$$\mathbf{a}' = (0, \gamma^2 v^2/r, 0). \tag{12.29}$$

REMARK: We can also arrive at the two factors of γ in $\mathbf{a}'$ by using a simple time-dilation argument. We have

$$a'_y = \frac{d^2y'}{d\tau^2} = \frac{d^2y'}{d(t/\gamma)^2} = \gamma^2\frac{d^2y}{dt^2} = \gamma^2\frac{v^2}{r}, \tag{12.30}$$

where we have used the fact that transverse lengths are the same in the two frames. ♣

12.6 The form of physical laws

One of the postulates of special relativity is that all inertial frames are equivalent. Therefore, if a physical law holds in one frame, then it must hold in all frames. Otherwise, it would be possible to differentiate between frames. As noted in the previous section, the statement "$\mathbf{f} = m\mathbf{a}$" cannot be a physical law. The two sides of the equation transform differently when going from one frame to another, so the statement cannot be true in all frames.

If a statement has any chance of being true in all frames, it must involve only 4-vectors. Consider a 4-vector equation (say, "$A = B$") which is true in frame S. Then if we apply to this equation a Lorentz transformation (call it $\mathcal{M}$) from S to another frame S', we have

$$\begin{aligned} A &= B, \\ \Longrightarrow \quad \mathcal{M}A &= \mathcal{M}B, \\ \Longrightarrow \quad A' &= B'. \end{aligned} \tag{12.31}$$

The law is therefore also true in frame S'.

Of course, there are many 4-vector equations that are simply not true (for example, $F = P$, or $2P = 3P$). Only a small set of such equations (for example, $F = mA$) correspond to the real world.

Physical laws may also take the form of scalar equations, such as $P \cdot P = m^2$. A scalar is by definition a quantity that is frame-independent (as we have shown the inner product to be). So if a scalar statement is true in one inertial frame, then it is true in all inertial frames. Physical laws may also be higher-rank "tensor" equations, such as arise in electromagnetism and general relativity. We won't discuss such things here, but suffice it to say that tensors may be thought of as things built up from 4-vectors. Scalars and 4-vectors are special cases of tensors.

All of this is exactly analogous to the situation in 3-D space. In Newtonian mechanics, $\mathbf{f} = m\mathbf{a}$ is a possible law, because both sides are 3-vectors. But $\mathbf{f} = m(2a_x, a_y, a_z)$ is not a possible law, because the right-hand side is not a 3-vector; it depends on which axis you label as the x-axis. Another example is the statement that a given stick has a length of 2 meters. That's fine, but if you say that the stick has an x-component of 1.7 meters, then this cannot be true in all frames.

> God said to his cosmos directors,
> "I've added some stringent selectors.
> One is the clause
> That your physical laws
> Shall be written in terms of 4-vectors."

12.7 Exercises

1. **Acceleration at rest**

   Show that the derivative of $v \equiv \sqrt{v_x^2 + v_y^2 + v_z^2}$ equals a_x, independent of how all the v_i's are changing, provided that $v_y = v_z = 0$ at the moment in question.

2. **Linear acceleration** $*$

   A particle's velocity and acceleration both point in the x-direction, with magnitudes v and $\dot{v}$, respectively (as measured in the lab frame). In the spirit of the example in Section 12.5.2, find the 3-acceleration and 4-acceleration in both the lab frame and the instantaneous rest frame of the particle. Verify that 3-accelerations are related according to eq. (12.25).

3. **Same speed** $*$

   Consider the setup in Problem 2. Given v, what should θ be so that the speed of one particle, as viewed by the other, is also v? Do your answers make sense for $v \approx 0$ and $v \approx c$?

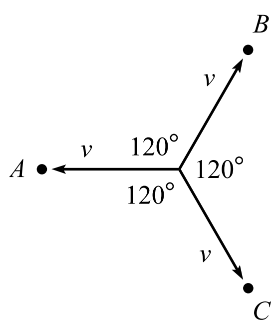

Figure 12.7

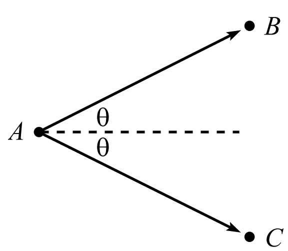

Figure 12.8

4. **Three particles** $**$

   Three particles head off with equal speeds v, at 120° with respect to each other, as shown in Fig. 12.7. What is the inner product of any two of the 4-velocities in any frame? Use your result to find the angle θ (see Fig. 12.8) at which two particles travel in the frame of the third.

5. **Doppler effect** $*$

   Consider a photon traveling in the x-direction. Ignoring the y and z components, and setting $c = 1$, the 4-momentum is (p, p). In matrix notation, what are the Lorentz transformations for the frames traveling to the left and to the right at speed v? What is the new 4-momentum of the photon in these new frames? Accepting the fact the a photon's energy is proportional to its frequency, verify that your results are consistent with the Doppler results in Section 10.6.1.

12.8 Problems

1. **Velocity addition**

   In A's frame, B moves to the right with speed u, and C moves to the left with speed v. What is the speed of B with respect to C? In other words, use 4-vectors to derive the velocity-addition formula.

2. **Relative speed** $*$

   In the lab frame, two particles move with speed v along the paths shown in Fig. 12.9. The angle between the trajectories is 2θ. What is the speed of one particle, as viewed by the other?

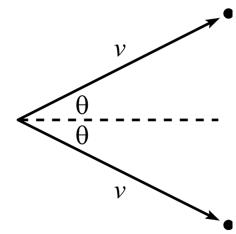

Figure 12.9

3. **Another relative speed** $*$

   In the lab frame, two particles, A and B, move with speeds u and v along the paths shown in Fig. 12.10. The angle between the trajectories is θ. What is the speed of one particle, as viewed by the other?

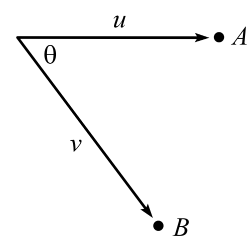

Figure 12.10

4. **Acceleration for linear motion** $*$

   A spaceship starts at rest with respect to frame S and accelerates with constant proper acceleration a. In Section 10.7, we showed that the speed of the spaceship with respect to S is given by $v(\tau) = \tanh(a\tau)$, where τ is the spaceship's proper time (and $c = 1$). Let V be the spaceship's 4-velocity, and let A be its 4-acceleration. In terms of the proper time τ,

   (a) Find V and A in frame S, by explicitly using $v(\tau) = \tanh(a\tau)$.

   (b) Write down V and A in the spaceship's frame, S'.

   (c) Verify that V and A transform like 4-vectors between the two frames.

12.9 Solutions

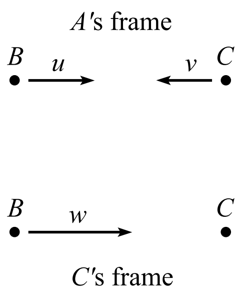

Figure 12.11

1. **Velocity addition**

   Let the desired speed of B with respect to C be w. See Fig. 12.11.

   In A's frame, the 4-velocity of B is $(\gamma_u, \gamma_u u)$, and the 4-velocity of C is $(\gamma_v, -\gamma_v v)$. We have suppressed the y and z components here.

   In C's frame, the 4-velocity of B is $(\gamma_w, \gamma_w w)$, and the 4-velocity of C is $(1, 0)$.

   The invariance of the inner product implies

$$\begin{aligned} (\gamma_u, \gamma_u u) \cdot (\gamma_v, -\gamma_v v) &= (\gamma_w, \gamma_w w) \cdot (1, 0) \\ \Longrightarrow \quad \gamma_u \gamma_v (1 + uv) &= \gamma_w \\ \Longrightarrow \quad \frac{1 + uv}{\sqrt{1-u^2}\sqrt{1-v^2}} &= \frac{1}{\sqrt{1-w^2}} \,. \end{aligned} \tag{12.32}$$

   Squaring and then solving for w gives

$$w = \frac{u+v}{1+uv} \,. \tag{12.33}$$

2. **Relative speed**

   In the lab frame, the 4-velocities of the particles are (suppressing the z component)

$$(\gamma_v, \gamma_v v\cos\theta, \gamma_v v \sin\theta) \qquad \text{and} \qquad (\gamma_v, \gamma_v v\cos\theta, -\gamma_v v\sin\theta). \tag{12.34}$$

   Let w be the desired speed of one particle as viewed by the other. Then in the frame of one particle, the 4-velocities are (suppressing two spatial components)

$$(\gamma_w, \gamma_w w) \qquad \text{and} \qquad (1, 0), \tag{12.35}$$

   where we have rotated the axes so that the relative motion is along the x-axis in this frame. Since the 4-vector inner product is invariant under Lorentz transformations and rotations, we have (using $\cos 2\theta = \cos^2\theta - \sin^2\theta$)

$$\begin{aligned} (\gamma_v, \gamma_v v\cos\theta, \gamma_v v\sin\theta) \cdot (\gamma_v, \gamma_v v\cos\theta, -\gamma_v v\sin\theta) &= (\gamma_w, \gamma_w w)\cdot(1,0) \\ \Longrightarrow \quad \gamma_v^2 (1 - v^2\cos 2\theta) &= \gamma_w . \end{aligned} \tag{12.36}$$

   Using the definitions of the γ's, squaring, and solving for w gives

$$w = \sqrt{1 - \frac{(1-v^2)^2}{(1-v^2\cos 2\theta)^2}} = \frac{\sqrt{2v^2(1-\cos 2\theta) - v^4 \sin^2 2\theta}}{1 - v^2\cos 2\theta} \,. \tag{12.37}$$

   If desired, this can be rewritten (using some double-angle formulas) in the form,

$$w = \frac{2v\sin\theta\sqrt{1 - v^2\cos^2\theta}}{1 - v^2\cos 2\theta} \,. \tag{12.38}$$

   Remark: If $2\theta = 180^\circ$, then $w = 2v/(1+v^2)$, in agreement with the standard velocity-addition formula. And if $\theta = 0^\circ$, then $w = 0$, as should be the case. If θ is very small, then you can show $w \approx 2v\sin\theta/\sqrt{1-v^2}$, which is simply the relative speed in the lab frame, multiplied by the time dilation factor between the frames. (The particles' clocks run slow, and transverse distances don't change, so the relative speed is larger in a particle's frame.) ♣

3. **Another relative speed**

In the lab frame, the 4-velocities of the particles are (suppressing the z component)

$$V_A = (\gamma_u, \gamma_u u, 0) \qquad \text{and} \qquad V_B = (\gamma_v, \gamma_v v\cos\theta, -\gamma_v v\sin\theta). \tag{12.39}$$

Let w be the desired speed of one particle as viewed by the other. Then in the frame of one particle, the 4-velocities are (suppressing two spatial components)

$$(\gamma_w, \gamma_w w) \qquad \text{and} \qquad (1, 0), \tag{12.40}$$

where we have rotated the axes so that the relative motion is along the x-axis in this frame. Since the 4-vector inner product is invariant under Lorentz transformations and rotations, we have

$$\begin{aligned} (\gamma_u, \gamma_u u, 0)\cdot(\gamma_v, \gamma_v v\cos\theta, -\gamma_v v\sin\theta) &= (\gamma_w, \gamma_w w)\cdot(1,0) \\ \Longrightarrow \qquad \gamma_u\gamma_v(1 - uv\cos\theta) &= \gamma_w. \end{aligned} \tag{12.41}$$

Using the definitions of the γ's, squaring, and solving for w gives

$$w = \sqrt{1 - \frac{(1-u^2)(1-v^2)}{(1-uv\cos\theta)^2}} = \frac{\sqrt{u^2 + v^2 - 2uv\cos\theta - u^2v^2\sin^2\theta}}{1 - uv\cos\theta}. \tag{12.42}$$

You can check various special cases of this result.

4. **Acceleration for linear motion**

(a) Using $v(\tau) = \tanh(a\tau)$, we have $\gamma = 1/\sqrt{1-v^2} = \cosh(a\tau)$. Therefore,

$$V = (\gamma, \gamma v) = \big(\cosh(a\tau), \sinh(a\tau)\big), \tag{12.43}$$

where we have suppressed the two transverse components of V. We then have

$$A = \frac{dV}{d\tau} = a\big(\sinh(a\tau), \cosh(a\tau)\big). \tag{12.44}$$

(b) The spaceship is at rest in its instantaneous inertial frame, so

$$V' = (1, 0). \tag{12.45}$$

In the rest frame, we also have

$$A' = (0, a). \tag{12.46}$$

Equivalently, these are obtained by setting $\tau = 0$ in the results from part (a), because the spaceship hasn't started moving at $\tau = 0$, as is always the case in the instantaneous rest frame.

(c) The Lorentz transformation matrix from S' to S is

$$\mathcal{M} = \begin{pmatrix} \gamma & \gamma v \\ \gamma v & \gamma \end{pmatrix} = \begin{pmatrix} \cosh(a\tau) & \sinh(a\tau) \\ \sinh(a\tau) & \cosh(a\tau) \end{pmatrix}. \tag{12.47}$$

We must check that

$$\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} = \mathcal{M}\begin{pmatrix} V_0' \\ V_1' \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} A_0 \\ A_1 \end{pmatrix} = \mathcal{M}\begin{pmatrix} A_0' \\ A_1' \end{pmatrix}. \tag{12.48}$$

These are easily seen to be true.