Second-Order Anisotropic Hydrodynamics

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Motivation

- Relativistic fluid dynamics is used to model HIC
- There are large corrections to an ideal fluid due to the rapid longitudinal expansion
- The QGP is an anisotropic plasma
- Can then build in the anisotropies from the beginning to create a more reliable approximation scheme to the QGP matter
Heavy Ion Collision

- (a): Initial state; Incoming nuclei
- (b): Hard collisions
- (c): Equilibrating quarks and gluons, initially out of equilibrium; Hydrodynamic evolution
- (d): Hadron freezeout

Rapid longitudinal expansion (and relatively weaker transverse motion) of the quark gluon plasma causes momentum space anisotropy
Isotropic hydrodynamics

Canonical way to derive viscous hydrodynamics is to linearize around an isotropic equilibrium distribution function

\[ y - y_0 = \delta y \ll 1, \quad y_0 = \frac{u \cdot p}{T} - \frac{\mu}{T} \]

\[ f(y) = f_{\text{eq}}(y_0) + f_{\text{eq}} (1 - af_{\text{eq}}) \delta y + O(\delta y^2) \]

\[ \equiv f_0 + \delta f \]

- Particle momentum-space is approximated at leading-order by a sphere
- Isotropic energy-momentum tensor (ignoring bulk viscous pressure)

\[ T^{\mu\nu} = (\mathcal{E} + \mathcal{P}) u^\mu u^\nu - \mathcal{P} g^{\mu\nu} + \pi^{\mu\nu}, \quad R_{\pi}^{-1} = \sqrt{\pi^{\mu\nu} \pi_{\mu\nu}} / \mathcal{P} \ll 1 \]
Viscous hydrodynamic limitations

- Look at Navier-Stokes solution for insights on the momentum-space anisotropies

\[ \pi_{\text{NS}}^{\mu \nu} = \eta \nabla^{\langle \mu} u^{\nu \rangle} \]

\[ \mathcal{P}_\perp = \mathcal{P} + \frac{1}{2} (\pi_{\text{NS}}^{xx} + \pi_{\text{NS}}^{yy}) = \mathcal{P} + \frac{2\eta}{3\tau} \]

\[ \mathcal{P}_L = \mathcal{P} + \pi_{\text{NS}}^{zz} = \mathcal{P} - \frac{4\eta}{3\tau} \]

Can recast pressure ratio in terms of the temperature of the system

\[ \frac{\mathcal{P}_L}{\mathcal{P}_\perp} = \frac{3\tau T - 16\bar{\eta}}{3\tau T + 8\bar{\eta}} , \quad \bar{\eta} \equiv \eta/S \]

- Initial conditions at RHIC: \( T_0 = 400 \text{ MeV}, \tau_0 = 0.5 \text{ fm/c} \) gives \( \mathcal{P}_L/\mathcal{P}_\perp \geq 0.5 \)
- Initial conditions at LHC: \( T_0 = 600 \text{ MeV}, \tau_0 = 0.25 \text{ fm/c} \) gives \( \mathcal{P}_L/\mathcal{P}_\perp \geq 0.35 \)
- The longitudinal pressure can become negative for large values of \( \bar{\eta} \), early times, or low temperatures
Hydrodynamic expansion breaks down in far-from-equilibrium situations

- i.e. the first-order correction becomes of the order of the leading-order piece in the perturbative expansion

Generalized solution

\[ f(x, p) = f_0(x, p) \sum_{\ell, \alpha} a_\alpha(x) P^{(\ell)}_\alpha(p) \]

- \( f_0 \) is the LO approximation (arbitrary weight factor)
- In order to obtain the most rapid convergence, choose \( f_0 \) such that it is as close as possible to the exact solution \( f \)
- The choice of \( f_0 \) is guided by general insights into the properties of \( f \) for the problem at hand
Anisotropic expansion

- In HIC, rapid longitudinal expansion suggests to use an $f_0$ distorted along the $p_z$ (beam)-direction with azimuthal symmetry

$$\xi > 0 \implies P_L < P_\perp$$

- Expansion around a “local anisotropic equilibrium” distribution function

$$f(x, p) = f_{iso} \left( \frac{\sqrt{m^2 + p_\perp^2 + (1 + \xi(x)) p_z^2}}{\Lambda(x)} \right) + \delta \tilde{f}$$

- $\xi(x)$ is the anisotropy parameter
- $\Lambda$ is the effective transverse temperature

$$\xi = \frac{\langle p_\perp^2 \rangle}{2\langle p_\perp^2 \rangle} - 1$$
Hydrodynamic tensor decomposition for anisotropic systems

- Expansion around a spheroidal distribution function
  \[ f(x, p) = f_{RS} + \delta \tilde{f} \]
  leads to \( P_x = P_y \neq P_z \)
- Dissipative currents caused by spheroidal deformation of particle momentum-space are treated non-perturbatively
- \( \delta \tilde{f} \) gives rise to dissipative currents which account for viscous effects other than those included in \( \delta f = f_{aniso} - f_{eq} \)

\[
\begin{align*}
  j^\mu &= \mathcal{N}_{aniso} u^\mu + \tilde{\mathcal{V}}^\mu \\
  T^{\mu\nu} &= \left( \mathcal{E}_{aniso} + \mathcal{P}_\perp + \tilde{\mathcal{P}} \right) u^\mu u^\nu - \left( \mathcal{P}_\perp + \tilde{\mathcal{P}} \right) g^{\mu\nu} + (\mathcal{P}_L - \mathcal{P}_\perp) z^\mu z^\nu + \tilde{\Pi}^{\mu\nu}
\end{align*}
\]

R. Ryblewski and W. Florkowski, 1103.1260; M. Martinez, R. Ryblewski, and M. Strickland, 1204.1473
Quasi-thermodynamic quantities

\[ \mathcal{N}(\xi, \Lambda) = \int \frac{d^3p}{(2\pi)^3} f_{RS} = \mathcal{R}_0(\xi) \mathcal{N}_{\text{iso}}(\Lambda) \]

\[ \mathcal{E}(\xi, \Lambda) = T^{00} = \mathcal{R}(\xi) \mathcal{E}_{\text{iso}}(\Lambda) \]

\[ \mathcal{P}_\perp(\xi, \Lambda) = \frac{1}{2} (T^{xx} + T^{yy}) = \mathcal{R}_\perp(\xi) \mathcal{P}_{\text{iso}}(\Lambda) \]

\[ \mathcal{P}_L(\xi, \Lambda) = T^{zz} = \mathcal{R}_L(\xi) \mathcal{P}_{\text{iso}}(\Lambda) \]

\[ \mathcal{R}(\xi) = \frac{1}{2} \left( \frac{1}{1 + \xi} + \frac{\arctan \sqrt{\xi}}{\sqrt{\xi}} \right), \]

\[ \mathcal{R}_\perp(\xi) = \frac{3}{2\xi} \left( \frac{1 + (\xi^2 - 1) \mathcal{R}(\xi)}{\xi + 1} \right), \]

\[ \mathcal{R}_L(\xi) = \frac{3}{\xi} \left( \frac{(\xi + 1) \mathcal{R}(\xi) - 1}{\xi + 1} \right) \]

- The bulk quantities factorize into a product of two functions only in massless limit
- In the \( m \neq 0 \) case can define an “anisotropic equation of state”
LO aHydro: \((0+1)d\) case

M. Martinez, M. Strickland, 1007.0889

\(0^{th}\) moment: \(\partial_\mu j^\mu \neq 0\)

\[
\frac{1}{1 + \xi} \partial_\tau \xi - \frac{2}{\tau} - \frac{6}{\Lambda \partial_\tau \Lambda} = 2\Gamma \left[ 1 - R^{3/4}(\xi) \sqrt{1 + \xi} \right]
\]

\(1^{st}\) moment: \(\partial_\mu T^{\mu \nu} = 0\)

\[
\frac{R'(\xi)}{R(\xi)} \partial_\tau \xi + \frac{4}{p_{\text{hard}}} \partial_\tau p_{\text{hard}} = \frac{1}{\tau} \left[ \frac{1}{\xi (1 + \xi) R(\xi)} - \frac{1}{\xi} - 1 \right]
\]

Relaxation rate \(\Gamma\) determined by matching to viscous hydrodynamics for small \(\xi\)

\[
\Gamma = \frac{2}{\tau_\pi} = \frac{2T}{5\bar{\eta}} = \frac{2R^{1/4}(\xi) \Lambda}{5\bar{\eta}}
\]
Pressure anisotropy

\[ \frac{\eta}{S} = \frac{10}{4\pi} \]

\( \tau_0 = 0.2 \text{ fm/c} \)
\( T_0 = 350 \text{ MeV} \)
\( \xi_0 = 0 \)

M. Martinez and MS, Nuclear Physics A 848, 183 (2010).
Transverse expansion

M. Martinez, R. Ryblewski, and M. Strickland, 1204.1473

\[ 4\pi \eta / S = 1, \text{ Pb-Pb @ 2.76 TeV: } T_0 = 600 \text{ MeV, } \tau_0 = 0.25 \text{ fm/c, } b = 7 \text{ fm} \]
Macrosopic equations of motion: \((2+1)d\)

0\textsuperscript{th} moment

\[ \dot{N} = -N \theta - \partial_\mu \tilde{\mathcal{V}}^\mu + C \]

1\textsuperscript{st} moment

\[ \dot{E} + (E + P_\perp + \tilde{\Pi}) \theta + (P_L - P_\perp) \frac{u_0}{\tau} + u_\nu \partial_\mu \tilde{\pi}^{\mu\nu} = 0 \]

\[ (E + P_\perp + \tilde{\Pi}) \dot{u}_x + \partial_x (P_\perp + \tilde{\Pi}) + u_x (P_L - P_\perp) \frac{u_0 u_x}{\tau} - \Delta^{1\nu} \partial_\mu \tilde{\pi}^{\mu\nu} = 0 \]

\[ (E + P_\perp + \tilde{\Pi}) \dot{u}_y + \partial_y (P_\perp + \tilde{\Pi}) + u_y (P_L - P_\perp) \frac{u_0 u_y}{\tau} - \Delta^{2\nu} \partial_\mu \tilde{\pi}^{\mu\nu} = 0 \]

- Need more equations for \(\tilde{\mathcal{V}}^\mu, \tilde{\Pi},\) and \(\tilde{\pi}^{\mu\nu}\)
“Anisotropic tansport” equations

- $\delta \tilde{f}$ is treated perturbatively like in viscous hydro
- Expand in a complete, orthogonal basis of Denicol et al 1202.4551
- Results in generalized equations of motion
- Use 14-moment approximation to truncate expansion

\[
\dot{\hat{n}} = -\frac{\hat{\gamma}_r}{\tilde{\gamma}_r} \hat{n} + \frac{1}{\tilde{\gamma}_r} C_{r-1} + \mathcal{W}_r + U_{r}^{\mu\nu} \nabla_{\mu} u_{\nu} \\
+ \lambda_{r}^{r} \pi^{\mu\nu} \sigma_{\mu\nu} + \tau_{r}^{r} V \tilde{\nu} \hat{u}_{\mu} - \frac{1}{\tilde{\gamma}_r} \nabla_{\mu} \left( \tilde{\gamma}_{r-1} \tilde{V}^{\mu} \right) - \delta_{r}^{r} \pi \hat{\theta}
\]

\[
\dot{\tilde{V}}^{\langle \mu \rangle} = -\frac{\hat{\gamma}_r}{\tilde{\gamma}_r} \tilde{V}^{\mu} + \frac{1}{\tilde{\gamma}_r} C_{r-1}^{\mu} + \mathcal{Z}_{r}^{\mu} - \tilde{V}^{\nu} \omega_{\nu}^{\mu} + \delta_{VV} \tilde{V}^{\mu} \theta - \Delta_{r}^{r} \lambda \frac{1}{\tilde{\gamma}_r} \nabla_{\nu} \left( \tilde{\gamma}_{r-1} \tilde{V}^{\nu} \pi_{r}^{\nu} \lambda \right) \\
+ \tau_{q}^{r} \pi^{\mu\nu} \hat{u}_{\nu} + \lambda_{VV} \tilde{V}_{\nu} \sigma_{\nu}^{\mu\nu} + \tau_{q}^{r} \pi \hat{n} \hat{u}_{\mu} + \ell_{q}^{r} \nabla^{\mu} \hat{n} + \hat{n} \xi^{\mu}
\]

\[
\dot{\pi}^{\langle \mu \nu \rangle} = -\frac{\hat{\gamma}_r}{\tilde{\gamma}_r} \pi^{\mu\nu} + \tau^{\langle \mu V \nu \rangle} + \frac{1}{\tilde{\gamma}_r} C_{r-1}^{\langle \mu \nu \rangle} + \mathcal{K}_{r}^{\mu\nu} + \mathcal{L}_{r}^{\mu\nu} + \mathcal{H}_{r}^{\mu\nu} \lambda \hat{z}_{\lambda} + \mathcal{Q}_{r}^{\mu\nu} \lambda_{\alpha} \nabla_{\lambda} u_{\alpha} + \lambda_{r}^{r} \nu \lambda_{\alpha} u_{\alpha} \nabla_{\lambda} z_{\alpha}
\\
- 2\lambda_{r}^{r} \pi_{\alpha}^{\mu} \nu_{\alpha} + 2\pi \lambda^{\mu} \omega^{\mu} + 2\lambda_{\pi}^{r} \pi \hat{n} \sigma^{\mu\nu} + 2\lambda_{r}^{r} V^{\langle \mu \nu \rangle} \tilde{V}^{\nu} + 2\tau_{\pi V}^{r} \tilde{V}^{\langle \mu \nu \rangle} - 2\delta_{\pi}^{r} \pi^{\mu\nu} \theta.
\]
\[(2+1)\text{d aHydro equations}\]

\[
\frac{\dot{\xi}}{1+\xi} - 6 \frac{\dot{\Lambda}}{\Lambda} - 2\theta = 2\Gamma \left( 1 - \sqrt{1+\xi} \mathcal{R}^{3/4}(\xi) \right)
\]

\[
\mathcal{R}' \dot{\xi} + 4\mathcal{R} \frac{\dot{\Lambda}}{\Lambda} = - \left( \mathcal{R} + \frac{1}{3} \mathcal{R}_\perp \right) \theta_\perp - \left( \mathcal{R} + \frac{1}{3} \mathcal{R}_L \right) \frac{u_0}{\tau} + \frac{\tilde{\pi}^{\mu\nu} \sigma_{\mu\nu}}{\mathcal{E}_0(\Lambda)}
\]

\[
[3\mathcal{R} + \mathcal{R}_\perp] \dot{u}_\perp = -\mathcal{R}_\perp \partial_\perp \xi - 4\mathcal{R}_\perp \frac{\partial_\perp \Lambda}{\Lambda} - u_\perp \left( \mathcal{R}' \dot{\xi} + 4\mathcal{R}_\perp \frac{\dot{\Lambda}}{\Lambda} \right)
\]

\[
[3\mathcal{R} + \mathcal{R}_\perp] u_\perp \dot{\phi}_u = -\mathcal{R}_\perp D_\perp \xi - 4\mathcal{R}_\perp \frac{D_\perp \Lambda}{\Lambda} - \frac{3}{\mathcal{E}_0(\Lambda)} \left( \frac{u_\perp \Delta_\nu^1 + u_\perp \Delta_\nu^2}{u_\perp} \right) \partial_\mu \tilde{\pi}^{\mu\nu}
\]

\[
\dot{\tilde{\pi}}^{\mu\nu} = -2\dot{u}_\alpha \tilde{\pi}^{\alpha(\mu} u^{\nu)} - \Gamma \left[ (\mathcal{P} - \mathcal{P}_\perp) \Delta^{\mu\nu} + (\mathcal{P}_L - \mathcal{P}_\perp) z^{\mu} z^{\nu} + \tilde{\pi}^{\mu\nu} \right] + \mathcal{K}_0^{\mu\nu} + \mathcal{L}_0^{\mu\nu}
\]

\[
+ \mathcal{H}_0^{\mu\nu\lambda} \dot{z}_{\lambda} + \mathcal{Q}_0^{\mu\nu\lambda} \nabla_{\lambda} u_\alpha + \mathcal{X}_0^{\mu\nu\lambda} u_\alpha \nabla_{\lambda} z_{\alpha} - 2\lambda_0^{0\pi\pi} \tilde{\pi}^{\lambda(\mu} \sigma^{\nu)}_{\lambda} + 2\tilde{\pi}^{\lambda(\mu} \omega^{\nu)}_{\lambda} - 2\delta^{0\pi\pi} \tilde{\pi}^{\mu\nu} \theta,
\]
Testing vaHydro

- Is there a way to test the accuracy of various approximation methods?

- Exact (numerical) solution to the Boltzmann equation in RTA exits for \((0+1)d\) systems [W. Florkowski, R. Ryblewski, M. Strickland 1304.0665, 1305.7234]

- Relaxation rate is determined by matching at asymptotically late time to the exact solution

\[
\Gamma = \frac{\mathcal{R}^{1/4}(\xi) \Lambda}{5\bar{\eta}}
\]
Pressure anisotropy

\[ P_L/P_T \]

\[ x_0 = 0, 4\pi\eta/S = 1, T_0 = 0.6 \text{ GeV} \]

\[ \xi_0 = 0, 4\pi\eta/S = 3, T_0 = 0.6 \text{ GeV} \]

\[ \xi_0 = 0, 4\pi\eta/S = 10, T_0 = 0.6 \text{ GeV} \]

\[ \xi_0 = 0, 4\pi\eta/S = 100, T_0 = 0.6 \text{ GeV} \]

\[ x_0 = 10, 4\pi\eta/S = 1, T_0 = 0.6 \text{ GeV} \]

\[ \xi_0 = 10, 4\pi\eta/S = 3, T_0 = 0.6 \text{ GeV} \]

\[ \xi_0 = 10, 4\pi\eta/S = 10, T_0 = 0.6 \text{ GeV} \]

\[ \xi_0 = 10, 4\pi\eta/S = 100, T_0 = 0.6 \text{ GeV} \]

\[ x_0 = 100, 4\pi\eta/S = 1, T_0 = 0.6 \text{ GeV} \]

\[ \xi_0 = 100, 4\pi\eta/S = 3, T_0 = 0.6 \text{ GeV} \]

\[ \xi_0 = 100, 4\pi\eta/S = 10, T_0 = 0.6 \text{ GeV} \]

\[ \xi_0 = 100, 4\pi\eta/S = 100, T_0 = 0.6 \text{ GeV} \]
Relative error of pressure ratio
Relative error of effective temperature
Particle production

![Graph](image)

- Exact Solution
- vaHydro
- aHydro
- 3rd-order hydro
- 2nd-order hydro
The anisotropic hydrodynamics framework is a more efficient way to solve the relativistic hydrodynamics equations for HIC.

Second-order anisotropic hydrodynamics allows for corrections to the spheroidal form.

Expect it to improve the validity of viscous hydrodynamics for HIC especially at early times, for large $\eta/S$, or near the transverse...